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The associated graded algebras of Brauer graph algebras II: Infinite representation type

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ABSTRACT

Let G be a Brauer graph and A the associated Brauer graph algebra. Denote by $\mathrm{gr}(A)$ the graded algebra associated with the radical filtration of A. The question when $\mathrm{gr}(A)$ is of finite representation type was answered in a previous paper. In the present paper, we characterize when $\mathrm{gr}(A)$ is domestic in terms of the associated Brauer graph G.

1. Introduction

This is a continuation of our study on the associated graded algebras of Brauer graph algebras in [10]. Since last paper has determined the finite representation type of this class of algebras, we focus in the present paper on the infinite representation type of them. In particular, we will characterize the domestic associated graded algebras of Brauer graph algebras.

Brauer graph algebras are finite dimensional algebras and originate in the modular representation theory of finite groups. They are defined by combinatorial data based on graphs: underlying every Brauer graph algebra is a finite graph with a cyclic orientation of the edges at every vertex and a multiplicity function.

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At least over an algebraically closed field, the class of Brauer graph algebras coincides with the class of symmetric special biserial algebras. For the representation theory of Brauer graph algebras, we refer the reader to the survey article [12].

The idea of associating a finite dimensional algebra A to a graded algebra (denoted by $\mathrm{gr}(A)$) with the radical filtration of A is not rare in representation theory (see for example, [3,11]). For a finite dimensional algebra A defined by quiver with relations, $\mathrm{gr}(A)$ often appears as a degeneration of A. The notion of degeneration comes from the geometric representation theory of algebras. It is known that if Λ_0 is a degeneration of some algebra Λ_1 and Λ_0 is representation-finite (resp. tame), then Λ_1 is also representation-finite (resp. tame) (see [7,8]). However, the representation type of Λ_0 is usually more complicated than that of Λ_1. In [10], we initiated the study on comparing the representation theory of $\mathrm{gr}(A)$ and that of A in case that A is a Brauer graph algebra. We have characterized all the algebras $\mathrm{gr}(A)$ which are of finite representation type and described the relationship between the Auslander-Reiten quivers of $\mathrm{gr}(A)$ and A in this case.

A Brauer graph algebra A is a self-injective (even symmetric) special biserial algebra; the associated graded algebra $\mathrm{gr}(A)$ is usually not self-injective. Nevertheless, $\mathrm{gr}(A)$ is still a special biserial algebra. Thus, both A and $\mathrm{gr}(A)$ have tame representation type. To describe the tameness more precisely, one needs the notions of domestic and polynomial growth. The relationship between these notions are: domestic $\Longrightarrow$ polynomial growth $\Longrightarrow$ tame (cf. Section 2.1). Bocian and Skowroński have characterized when a Brauer graph algebra A is domestic in [2]. In the present paper, we characterize when the associated graded algebra $\mathrm{gr}(A)$ is domestic.

To state our main result precisely, let us first introduce some notations.

Definition 1.1 *(See [10, Definition 2.4]).* Let G be a Brauer graph. For each vertex v, we denote by $m(v)$ the multiplicity of v and by $\mathrm{val}(v)$ the valency of v, with the convention that a loop is counted twice in $\mathrm{val}(v)$. Moreover, if $\mathrm{val}(v) = 1$, we denote by v' the unique vertex adjacent to v. For each vertex v in G, we define the graded degree $\mathrm{grd}(v)$ as follows.

$$\mathrm{grd}(v) = \begin{cases} m(v)\mathrm{val}(v), & \text{if } m(v)\mathrm{val}(v) > 1, \\ m(v')\mathrm{val}(v'), & \text{if } m(v)\mathrm{val}(v) = 1. \end{cases}$$

Definition 1.2 *(Compare with [10, Definition 2.12]).* Let G be a Brauer graph.

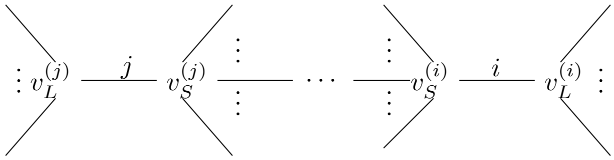
(1) If $u \overset{i}{\text{——}} v$ is an edge in G, we write the subgraph of G by removing the edge i as follows: $G \setminus i = G_{i,u} \bigcup G_{i,v}$, where $G_{i,u}$ (resp. $G_{i,v}$) is the connected branch of $G \setminus i$ containing the vertex u (resp. v). (Note that it may happen that $G_{i,u} = G_{i,v}$.) Moreover, we denote the set of vertices in $G_{i,u}$ (resp. $G_{i,v}$) by $V(G_{i,u})$ (resp. $V(G_{i,v})$).
(2) An unbalanced edge in G is defined to be an edge associated with two vertices with different graded degrees. For any unbalanced edge $v_S \overset{i}{\text{——}} v_L$ with $\mathrm{grd}(v_S) < \mathrm{grd}(v_L)$ in G, we write the subgraph of G by removing the edge i as follows: $G \setminus i = G_{i,L} \bigcup G_{i,S}$, where $G_{i,L}$ (resp. $G_{i,S}$) is the connected branch of $G \setminus i$ containing the vertex v_L (resp. v_S).

Definition 1.3 *(See Section 3 for the details).* Let G be a Brauer graph.

(1) A walk (for the notion of a walk in G, see Definition 3.1 below) v_1 —— v_2 —— $\cdots$ —— v_k from v_1 to v_k in a Brauer graph G is called degree decreasing if $\mathrm{grd}(v_1) \geq \mathrm{grd}(v_2) \geq \cdots \geq \mathrm{grd}(v_k)$.
(2) Suppose that G is a Brauer tree with an exceptional vertex v_0 of multiplicity m_0.
    (2.1) κ_0 is defined to be the number of unbalanced edges $v_S \overset{i}{\text{——}} v_L$ in G such that the exceptional vertex v_0 is a vertex in $G_{i,S}$.

(2.2) For any vertices u and v in G, $d_G(u,v)$ is defined to be the number of edges in the unique walk from u to v.

(2.3) Given two unbalanced edges $v_S^{(i)} \overset{i}{\text{——}} v_L^{(i)}$ and $v_S^{(j)} \overset{j}{\text{——}} v_L^{(j)}$ in G, we call the set $\{i,j\}$ an unbalanced edge pair if j is an edge in $G_{i,S}$ and $d_G(v_S^{(j)}, v_S^{(i)}) + 1 = d_G(v_L^{(j)}, v_S^{(i)})$.



Let κ_1 be the number of unbalanced edge pairs in G.

With the notations above, we have the following main result. One interesting point in this result is that, as in the finite representation type situation (see [10]), the graded degree function plays a key role in controlling the domestic type of $\mathrm{gr}(A)$.

Theorem 1.4 *(See Theorem 3.18). Let A be the Brauer graph algebra associated with a Brauer graph $G = (V(G), E(G), m)$ and $\mathrm{gr}(A)$ the graded algebra associated with the radical filtration of A, where $V(G)$ is the vertex set, $E(G)$ is the edge set and m is the multiplicity function of G. Then $\mathrm{gr}(A)$ is of polynomial growth if and ony if $\mathrm{gr}(A)$ is domestic.*

Furthermore, we have

(*I*) $\mathrm{gr}(A)$ *is 1-domestic if and only if one of the following holds:*
- (1) *G is a Brauer tree with an exceptional vertex v_0 of multiplicity m_0 such that $\kappa_0(m_0-1)+\kappa_1 = 1$.*
- (2) *G is a tree and there exist two distinct vertices w_0, w_1, such that the following conditions hold:*
  - (2.1) *$m(w_0) = m(w_1) = 2$ and $m(v) = 1$ for $v \neq w_0, w_1$,*
  - (2.2) *$\mathrm{grd}(w_0) = \mathrm{grd}(w_1)$,*
  - (2.3) *Any walk from w_0 (or from w_1) is degree decreasing.*
- (3) *G is a graph with a unique cycle of odd length and $m(v) = 1$ for all $v \in V(G)$, and satisfies the following conditions:*
  - (3.1) *$\mathrm{grd}(u) = \mathrm{grd}(v)$ for any two vertices u and v in the unique cycle,*
  - (3.2) *Any walk from any vertex in the unique cycle is degree decreasing.*

(*II*) $\mathrm{gr}(A)$ *is 2-domestic if and only if G satisfies the following conditions:*
- (1) *G is a graph with a unique cycle of even length and $m(v) = 1$ for all $v \in V(G)$,*
- (2) *$\mathrm{grd}(u) = \mathrm{grd}(v)$ for any two vertices u and v in the unique cycle,*
- (3) *Any walk from any vertex in the unique cycle is degree decreasing.*

(*III*) $\mathrm{gr}(A)$ *is not n-domestic for $n \geq 3$.*

This paper is organized as follows. In Section 2, we recall various definitions and known facts needed in this paper, including representation type of finite dimensional algebras, special biserial algebras and string algebras, Brauer graph algebras and their associated graded algebras. In Section 3, we first introduce the notions of $\star$-condition and unbalanced edge pair and prove some preliminary results; then we state our main result and its consequences. The proof of main result is based on careful analyses in different cases according to the shapes of Brauer graphs; the detailed proofs and examples of three main cases are filled in Section 4-6 respectively.

2. Preliminaries

Throughout this paper, we fix an algebraically closed field k. Unless otherwise stated, all algebras will be finite dimensional k-algebras, and all their modules will be finite dimensional left modules. For a k-algebra A, we denote by $\mathrm{rad}(A)$ the Jacobson radical of A. For an A-module M, we denote by $\mathrm{soc}(M)$ and $\mathrm{rad}(M)$ the socle and the radical of M, respectively. The length of a module M is denoted by $\ell(M)$, it means the number of composition factors in any composition series of M.

2.1. Representation type of finite dimensional algebras

We recall the various notions on representation types of finite dimensional algebras and their relations from the textbook [13, Section XIX.3].

Let A be a finite dimensional k-algebra. We say that A is of finite representation type, if there are only finitely many non-isomorphic indecomposable A-modules.

Let $k[t]$ be the polynomial algebra in one variable over k. We say that A is of tame representation type, if for any dimension d, there exists a finite number of A-$k[t]$-bimodules Q_i, for $1 \leq i \leq n_d$, which are finitely generated and free as right $k[t]$-modules such that all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form $Q_i \otimes_{k[t]} k[t]/(t-\lambda)$ for some $\lambda \in k$ and some i. For each d, let $\mu_A(d)$ be the least number of such A-$k[t]$-bimodules. We say that A is of polynomial growth type if there exists a positive integer m such that $\mu_A(d) \leq d^m$ for all $d \geq 2$; A is of finite growth type (or equivalently, domestic) if $\mu_A(d) \leq m$ for some positive integer m and for all $d \geq 1$ and A is n-domestic (or n-parametric) if n is the least such integer m.

Clearly every domestic algebra is of polynomial growth. In other words, if an algebra is not of polynomial growth, then the algebra is nondomestic. For the examples of nondomestic algebras of polynomial growth, we refer the reader to [14].

It is well known that an algebra of infinite representation type that is not of tame representation type is of wild representation type, however, our study does not involve the wild representation type.

2.2. Special biserial algebras and string algebras

These algebras are defined by quivers with relations. For more details on these algebras, we refer to [1], [5], and [12].

For a quiver Q, we denote by Q_0 and Q_1 its vertex set and arrow set respectively. We write a path p in a quiver from right to left and denote by $s(p)$ and $t(p)$ the start and the end of p, respectively. The length of a path is defined in an obvious way. As usual, the trivial path at a vertex i is denoted by e_i.

Definition 2.1. A finite dimensional k-algebra A is called special biserial if there is a quiver Q and an admissible ideal I in kQ such that A is Morita equivalent to kQ/I and such that kQ/I satisfies the following conditions:
(1) At every vertex v in Q there are at most two arrows starting at v and there are at most two arrows ending at v.
(2) For every arrow α in Q, there exists at most one arrow β such that $\beta\alpha \notin I$ and there exists at most one arrow γ such that $\alpha\gamma \notin I$.

A special biserial algebra A is called a string algebra if the defining ideal I is generated by paths.

Given a special biserial algebra $A = kQ/I$, we can associate a string algebra $\bar{A}$ as follows. Set

$$L := \{i \in Q_0 \mid Ae_i \text{ is an injective and not uniserial module}\},$$

$$S_0 := \bigoplus_{i \in L} \mathrm{soc}(Ae_i),$$

where Ae_i denotes the indecomposable projective A-module at vertex i. Then S_0 is an ideal of A and the quotient algebra $\bar{A} = A/S_0$ is a string algebra. Note that the operation $\overline{(\cdot)}$ preserves representation type (see the lemma on separation in [4]) and we can reconstruct the AR-quiver of A from the AR-quiver of $\overline{A}$ easily (cf. [5, Section II.1.3]).

Suppose now that $A = kQ/I$ is a string algebra. For an arrow $\beta \in Q_1$, we denote by β^{-1} the formal inverse of β and set $s(\beta^{-1}) = t(\beta)$, $t(\beta^{-1}) = s(\beta)$, $(\beta^{-1})^{-1} = \beta$. For convenience, the formal inverse of an arrow will be called an inverse arrow. A word of length n is defined by a sequence $c_n \dots c_2 c_1$, where $c_i \in Q_1$ or $c_i^{-1} \in Q_1$, and where $t(c_i) = s(c_{i+1})$ for $1 \leq i \leq n-1$. We define

$$s(c_n \dots c_2 c_1) = s(c_1),\ t(c_n \dots c_2 c_1) = t(c_n),$$

and

$$(c_n \dots c_2 c_1)^{-1} = c_1^{-1} c_2^{-1} \dots c_n^{-1}.$$

For every vertex v in Q, there is an empty word 1_v of length 0 such that $t(1_v) = s(1_v) = v$ and $1_v^{-1} = 1_v$. Suppose that a word $C := c_n \dots c_2 c_1$ satisfies $s(C) = t(C)$, we define a rotation of C as a word of the form $c_i \dots c_1 c_n \dots c_{i+1}$. The product of two words is defined by placing them next to each other, provided that the resulting sequence is a word.

A word C is called a string provided either $C = 1_v$ for some vertex v in Q or $C = c_n \dots c_2 c_1$ satisfying $c_{i+1} \neq c_i^{-1}$ for $1 \leq i \leq n-1$, and no subword (or its inverse) of C belongs to the ideal I. We say that a string $C = c_n \dots c_2 c_1$ with $n \geq 1$ is directed if all c_i are arrows, and C is inverse if all c_i are inverse arrows. A string C of positive length is called a band if all powers of C are strings and C is not a power of a string of smaller length. Note that a band must contain both arrows and inverse arrows.

On the set of strings, we consider two equivalence relations. Firstly, $\sim$ denotes the relation which identifies C and C^{-1}; and secondly, we define $\sim_A$ to be the equivalence relation which identifies each word with its rotations and their inverses. Let $\mathrm{St}(A)$ (or simply St) be a set of representatives of strings in A under $\sim$, and let $\mathrm{Ba}(A)$ (or simply Ba) be the set of representatives of bands under $\sim_A$. In the following, we call a subword of a string a substring.

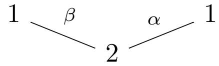
It is well known that every indecomposable module over a string algebra is either a string module or a band module. For each element C in $\mathrm{St}(A)$, there is a unique string A-module $M(C)$ up to isomorphism. For each element B in $\mathrm{Ba}(A)$ and for any finite dimensional indecomposable $k[x, x^{-1}]$-module $M = (V, \varphi)$ (where V is a n-dimensional k-vector space and φ is an invertible linear endomorphism of V), there is a band module $M(B, n, \varphi)$ corresponding to B and M. For a detailed explanation of $M(C)$ and $M(B, n, \varphi)$, we refer the reader to [1, p.160-161].

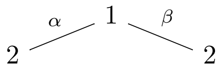
Example 2.2. Let $A = kQ$ be the Kronecker algebra defined by the following quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2\ ,$$

Then A is a string algebra and we can choose St and Ba as follows.

$$\mathrm{St} = \{1_1, 1_2, \alpha, \beta, \beta^{-1}\alpha, \alpha\beta^{-1}, \beta\alpha^{-1}\beta, \alpha\beta^{-1}\alpha, \cdots\}, \quad \mathrm{Ba} = \{\beta^{-1}\alpha\}.$$

The string module $M(\beta^{-1}\alpha)$ has the Loewy diagram 

, and the string module $M(\alpha\beta^{-1})$ has the Loewy diagram 

. The band module $M(\beta^{-1}\alpha, 2, \varphi)$ defined by the band $\beta^{-1}\alpha$ and the $k[x, x^{-1}]$-module $(k^2, \varphi = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix})$ corresponds to the representation

$$k^2 \underset{\varphi^{-1}}{\overset{I_2}{\rightrightarrows}} k^2,$$

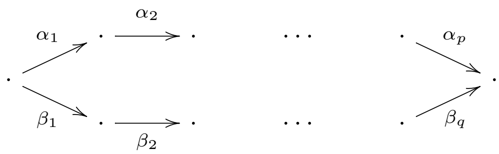
where $0 \neq \lambda \in k$ and I_2 denotes the 2×2 identity matrix.

For the representation types of special biserial algebras, there is the following theorem.

Theorem 2.3 *([5, II.3.1 and II.8.1]).*

(1) *Any special biserial algebra A is tame.*
(2) *A string algebra A is of finite representation type if and only if there is no band in A.*

The representation type and Auslander-Reiten quivers for self-injective special biserial algebras are well studied by Erdmann and Skowroński in [6]. Before stating their results, we recall some notions. For any algebra A, we denote by Γ_A the Auslander-Reiten quiver of A and by ${}_s\Gamma_A$ the stable Auslander-Reiten quiver of A. For the shapes of the translation quivers $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty/ < \tau^n >$, $\tilde{A}_{p,q}$, we refer to [9]. By $\tilde{A}_{p,q}$ we denote the following orientation of the quiver with underlying extended Dynkin diagram of type $\tilde{A}_{p+q-1}$:



Theorem 2.4 *([6, Theorem 2.1]). Let $A = kQ/I$ be a self-injective special biserial algebra. The following are equivalent:*

(1) *${}_s\Gamma_A$ has a component of the form $\mathbb{Z}\tilde{A}_{p,q}$.*
(2) *${}_s\Gamma_A$ is infinite but has no component of the form $\mathbb{Z}A_\infty^\infty$.*
(3) *There are positive integers m, p, q such that ${}_s\Gamma_A$ is a disjoint union of m components of the form $\mathbb{Z}\tilde{A}_{p,q}$, m components of the form $\mathbb{Z}A_\infty/ < \tau^p >$, m components of the form $\mathbb{Z}A_\infty/ < \tau^q >$ and infinitely many components of the form $\mathbb{Z}A_\infty/ < \tau >$.*
(4) *All but a finite number of components of Γ_A are of the form $\mathbb{Z}A_\infty/ < \tau >$.*
(5) *The number of primitive walks in A is a positive integer.*
(6) *A is representation-infinite domestic.*
(7) *A is representation-infinite of polynomial growth.*

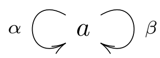
Theorem 2.5 *([6, Theorem 2.2]). Let $A = kQ/I$ be a self-injective special biserial algebra. The following are equivalent:*

(1) *${}_s\Gamma_A$ has a component of the form $\mathbb{Z}A_\infty^\infty$.*

(2) *${}_s\Gamma_A$ has infinitely many (regular) components of the form $\mathbb{Z}A_\infty^\infty$.*
(3) *${}_s\Gamma_A$ is a disjoint union of a finite number of components of the form $\mathbb{Z}A_\infty / < \tau^n >$ with $n > 1$, infinitely many components of the form $\mathbb{Z}A_\infty / < \tau >$ and infinitely many components of the form $\mathbb{Z}A_\infty^\infty$.*
(4) *A has infinitely many primitive walks.*
(5) *A is not of polynomial growth.*

Remark 2.6. For the definition of primitive walks (= primitive V-sequences) in a special biserial algebra A, we refer to [15, Section 2]. In fact, the primitive walks in A precisely correspond to the bands in $\overline{A}$.

Example 2.7. Let $A = kQ/I$ be the self-injective special biserial algebra defined by the following quiver



and the admissible ideal I generated by α^2, β^2 and $\alpha\beta - \beta\alpha$. We can choose Ba for $\overline{A}$ as follows.

$$\mathrm{Ba} = \{\beta^{-1}\alpha\}.$$

Let Q' be the following quiver:

$$1 \underset{\zeta_2}{\overset{\zeta_1}{\rightrightarrows}} 2 \ .$$

We have a quiver homomorphism u from Q' to Q as follows.

$$u(1) = u(2) = a, \ u(\zeta_1) = \alpha, \ u(\zeta_2) = \beta.$$

By [15, Section 2], we have that u is a primitive walk in A, which corresponds to the band $\beta^{-1}\alpha$ in $\overline{A}$.

We would like to state a general result on domestic string algebras, which should be well-known but we could not find a proof in the literature. We are grateful to Nengqun Li for helpful discussion on this result.

Proposition 2.8. *Let $A = kQ/I$ be a string algebra and n a positive integer. Then A is n-domestic if and only if the cardinality of* $\mathrm{Ba}(A)$ *is n.*

Proof. First notice that since the number of string modules of a given dimension is finite, it suffices to consider band modules when we consider the representation type of a representation-infinite string algebra.

It is enough to show that the cardinality of $\mathrm{Ba}(A)$ is n implies that A is n-domestic. We first show that A is $\leq n$-domestic. Let $\mathrm{Ba}(A) = \{b_1, \cdots, b_n\}$. Then we can construct (band-like) A-$k[t]$-bimodules $M_1, \cdots, M_n$ which are finitely generated free as right $k[t]$-modules corresponding to the bands $b_1, \cdots, b_n$, respectively (cf. [5, Example I.4.3]). Then each band module of A is isomorphic to some $M_i \otimes_{k[t]} V$ for some finite dimensional indecomposable $k[t]$-module V. This shows that A is $\leq n$-domestic.

Next we show that A is $\geq n$-domestic. Suppose that $\mathrm{Ba}(A) = \{b_1, \cdots, b_n\}$. Choose a positive integer d such that the length of b_i divides d for each $1 \leq i \leq n$. Then for each i we can construct a family of band modules $\{W_{i,\lambda} \mid \lambda \in k^*\}$ of dimension d, using the band b_i, so that A is $\geq n$-domestic. □

2.3. Brauer graph algebras and their associated graded algebras

In this subsection, we briefly recall some notions and results on Brauer graphs, Brauer graph algebras and their associated graded algebras. For more details and examples, we refer to [12, Section 2] and [10, Section 2].

Recall that a Brauer graph is denoted by $G = (V(G), E(G), m, \mathfrak{o})$, where $V(G)$ is the vertex set, $E(G)$ is the edge set, m is the multiplicity function, and $\mathfrak{o}$ is the orientation of G (that is, for each vertex v, there is a multiplicity $m(v) \in \mathbb{Z}_{>0}$ and a cyclic ordering of the edges incident to v). We always assume that the associated graph G is connected and often leave out the symbol $\mathfrak{o}$.

A Brauer tree is a Brauer graph $G = (V(G), E(G), m)$ such that $(V(G), E(G))$ is a tree and $m(v) = 1$ for all but at most one $v \in V(G)$. In this case we always choose a specified vertex v_0 (if $m(v) > 1$ then we choose $v_0 = v$), called the exceptional vertex, whose multiplicity will be denoted by m_0.

In a Brauer graph $G = (V(G), E(G), m)$, we denote by $\mathrm{val}(v)$ the valency or the ordinary degree of a vertex $v \in V(G)$; it is defined to be the number of edges in G incident to v, with the convention that a loop is counted twice in $\mathrm{val}(v)$. An edge $i \in E(G)$ is said to be truncated at a vertex v if i is incident to v such that $m(v)\mathrm{val}(v) = 1$.

The Brauer graph algebra A associated with a Brauer graph $G = (V(G), E(G), m)$ has the form kQ/I, where the vertex set Q_0 of Q is identified with the edge set $E(G)$ of G, and the arrow set Q_1 of Q is determined by the orientation of G (see [10, Subsection 2.1] for the details). Note that there are at most two arrows starting and ending at every vertex of Q. Every vertex $v \in V(G)$ such that $m(v)\mathrm{val}(v) \geq 2$ (that is, v is not truncated), gives rise to a unique cycle C_v in Q, called a special cycle at v (note that $\mathrm{val}(v)$ is the number of arrows in C_v). If G contains no loops, then any special cycle in Q is a simple cycle (that is, a cycle with no repeated arrows and no repeated vertices). Let C_v be such a special cycle at v. Then if C_v is a representative in its cyclic permutation class such that $t(C_v) = i = s(C_v)$, $i \in Q_0$, we say that C_v is a special i-cycle at v. If a special i-cycle at v has starting arrow α, then we denote this special i-cycle at v by $C_v(\alpha)$. Note that if $i \in E(G)$ is not a loop, then the special i-cycle at v is unique and we simply write it by C_v.

The ideal I in kQ is generated by three types of relations:

Relation of the first type:

$$C_v(\alpha)^{m(v)} - C_{v'}(\alpha')^{m(v')},$$

for any $i \in Q_0$ and for any special i-cycles $C_v(\alpha)$ and $C_{v'}(\alpha')$ at v and v' respectively such that both v and v' are not truncated.

Relation of the second type:

$$\alpha C_v(\alpha)^{m(v)},$$

for any $i \in Q_0$, any $v \in V(G)$ and where $C_v(\alpha)$ is a special i-cycle at v with starting arrow α.

Relation of the third type:

$$\beta\alpha,$$

for any $\alpha, \beta \in Q_1$ such that $\beta\alpha$ is not a subpath of any special cycle except if $\beta = \alpha$ is a loop associated with a vertex v of valency one and multiplicity $m(v) > 1$.

It is well known that Brauer graph algebras coincide with symmetric special biserial algebras. From this point of view, Bocian and Skowroński give a characterization of the domestic Brauer graph algebras in [2].

Theorem 2.9 *(cf. [12, Corollary 2.9 and Theorem 5.1]). Let A be the Brauer graph algebra with a Brauer graph $G = (V(G), E(G), m)$, where $V(G)$ is the vertex set, $E(G)$ is the edge set and m is the multiplicity function of G. Then*

(a) *A is of finite representation type if and only if G is a Brauer tree.*
(b) *A is 1-domestic if and only if one of the following holds*

(1) *G is a tree with $m(v) = 2$ for exactly two vertices $v = w_0, w_1 \in V(G)$ and $m(v) = 1$ for all $v \in V(G)$, $v \neq w_0, w_1$.*
(2) *G is a graph with a unique cycle of odd length and $m(v) = 1$ for all $v \in V(G)$.*

(c) *A is 2-domestic if and only if G is a graph with a unique cycle of even length and $m(v) = 1$ for all $v \in V(G)$.*

(d) *There are no n-domestic Brauer graph algebras for $n \geq 3$.*

Note that if G is not one of the above mentioned cases $(a), (b), (c)$ in Theorem 2.9, then by Theorem 2.4, the corresponding Brauer graph algebra A is not of polynomial growth.

We now turn to the associated graded algebras of Brauer graph algebras. The notion of the graded algebra (denoted by $\mathrm{gr}(A)$) associated to a finite dimensional algebra A with the radical filtration of A plays an important role in the representation theory. For the definition and elementary properties of $\mathrm{gr}(A)$, we refer to [10, Subsection 2.2]. Recall from [10, Subsection 2.3] that, for a Brauer graph algebra $A = kQ/I$ associated with a Brauer graph G, the graded algebra $\mathrm{gr}(A)$ (associated with the radical filtration) of A has the same dimension with A and can be described by the same quiver and some modified relations. More precisely, $\mathrm{gr}(A) = kQ/I'$, where I' is an admissible ideal in kQ generated by relations of the second and the third types in I and modified relations of the first type in I. For a relation of the first type $C_v(\alpha)^{m(v)} - C_{v'}(\alpha')^{m(v')}$ in I, its modified relation is defined by the term of smaller length between $C_v(\alpha)^{m(v)}$ and $C_{v'}(\alpha')^{m(v')}$.

From the above description, we know that $\mathrm{gr}(A)$ is also special biserial (but not necessarily self-injective). Thus we can reduce the study on the representation types of A and $\mathrm{gr}(A)$ to that of their associated string algebras $\overline{A}$ and $\overline{\mathrm{gr}(A)}$. The string algebra $\overline{A}$ is defined by

$$\overline{A} = A/(\bigoplus_{i\in L} \mathrm{soc}(Ae_i)), \tag{2.1}$$

where

$$L = \{i \in Q_0 | \operatorname{rad}(Ae_i)/\operatorname{soc}(Ae_i) = V_{i,1} \oplus V_{i,2}, V_{i,1} \neq 0, V_{i,2} \neq 0\}.$$

For each $i \in L$, there is a relation $\rho_i = p_i - q_i$ of the first type in I, where the length of p_i is $\ell(V_{i,1}) + 1$, the length of q_i is $\ell(V_{i,2}) + 1$. Therefore $\overline{A}$ can be described by the same quiver Q and an admissible ideal I_1 in kQ, where I_1 is generated by the ideal I and new relations $\{p_i, q_i \mid i \in L\}$. Similarly, the string algebra $\overline{\mathrm{gr}(A)}$ is defined by

$$\overline{\mathrm{gr}(A)} = \mathrm{gr}(A)/(\bigoplus_{i\in L'} \mathrm{soc}(\mathrm{gr}(A)e_i)), \tag{2.2}$$

where

$$L' = \{i \in L | \ell(V_{i,1}) = \ell(V_{i,2})\}.$$

Note that for each $i \in L'$, there is a relation $\rho_i = p_i - q_i$ in I' such that p_i and q_i have the same length. Therefore $\overline{\mathrm{gr}(A)}$ can be described by the same quiver Q and an admissible ideal I_2 in kQ, where I_2 is generated by the ideal I' and new relations $\{p_i, q_i \mid i \in L'\}$.

As a conclusion, the four concerned algebras have the same quiver and the following displayed formulas:

$$A = kQ/I, \quad \overline{A} = kQ/I_1, \quad \mathrm{gr}(A) = kQ/I', \quad \overline{\mathrm{gr}(A)} = kQ/I_2.$$

In order to describe some relationships among these algebras, we further generalize some notions from [10].

Definition 2.10 *(Compare with [10, Definition 2.12]).* Let $G = (V(G), E(G), m)$ be a Brauer graph with graded degree function grd and $A = kQ/I$ the corresponding Brauer graph algebra. We identify Q_0 with $E(G)$ by the natural bijection between them.

(1) For an unbalanced edge $u \overset{i}{\text{——}} v$ in G, we denote the endpoints of i by $v_L^{(i)}, v_S^{(i)}$ with $\mathrm{grd}(v_L^{(i)}) > \mathrm{grd}(v_S^{(i)})$. Whenever the context is clear we will omit the superscript (i). Moreover, we define

$$n_i = \text{ the number of edges in } G_{i,S}, \tag{2.3}$$

where $G_{i,S}$ is the connected branch of $G \setminus i$ containing the vertex v_S.

(2) For an unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G, there is a relation of the first type $\rho_i = p_i - q_i$ in I, where $p_i = C_{v_S}^{m(v_S)}$, $q_i = C_{v_L}^{m(v_L)}$ are paths with lengths $\mathrm{grd}(v_S)$, $\mathrm{grd}(v_L)$ respectively. We define the following sets:

$$\mathbb{W} = \{i \in Q_0 | \operatorname{rad}(Ae_i)/\operatorname{soc}(Ae_i) = V_1 \oplus V_2, V_1 \neq 0, V_2 \neq 0, \ell(V_1) \neq \ell(V_2)\} \subseteq Q_0, \tag{2.4}$$

$$\mathbb{P} = \bigcup_{i \in \mathbb{W}} \{r_i | r_i \text{ is the longer path between } p_i \text{ and } q_i\}. \tag{2.5}$$

Note that the set of unbalanced edges is identified with $\mathbb{W}$ under the natural bijection between Q_0 and $E(G)$, and that $s(r_i) = t(r_i) = i$ for $r_i \in \mathbb{P}$.

By the definitions of $\overline{A}$ and $\overline{gr(A)}$, we have that $\overline{A}$ is a quotient algebra of $\overline{\mathrm{gr}(A)}$, that is, $\overline{A} \cong \overline{\mathrm{gr}(A)}/I_3$, where the ideal I_3 is the k-vector space with basis given by the paths in the set $\mathbb{P}$. In particular, any indecomposable $\mathrm{gr}(A)$-module that is not $\overline{\mathrm{gr}(A)}$-module is an indecomposable projective-injective $\mathrm{gr}(A)$-module.

For convenience, we record displayed formulas of the ideals I, I', I_1, I_2, I_3 in kQ (see [10, Subsection 3.2]):

$$R_1 := \{\text{Relation of the first type in } I\}, \quad I_0 := \langle \text{Relation of the second type or the third type in } I\rangle;$$
$$I = I_0 + \langle R_1 \rangle;$$
$$I' = I_0 + \langle p_i - q_i \in R_1 \mid i \in Q_0, i \notin \mathbb{W}\rangle + \langle q_i \mid i \in \mathbb{W}, p_i - q_i \in R_1, q_i \text{ is shorter than } p_i\rangle;$$
$$I_1 = I_0 + \langle p_i, q_i \mid i \in Q_0, p_i - q_i \in R_1\rangle;$$
$$I_2 = I_0 + \langle p_i, q_i \mid i \in Q_0, i \notin \mathbb{W}, p_i - q_i \in R_1\rangle + \langle q_i \mid i \in \mathbb{W}, p_i - q_i \in R_1, q_i \text{ is shorter than } p_i\rangle;$$
$$I_3 = \langle r_i \in \mathbb{P} \mid i \in \mathbb{W}, p_i - q_i \in R_1\rangle = k\text{-vector space with basis } \{r_i \in \mathbb{P} \mid i \in \mathbb{W}\}.$$

The following proposition describes when $\mathrm{gr}(A)$ and A are isomorphic.

Proposition 2.11 *([10, Proposition 2.13]). Let $A = kQ/I$ be a Brauer graph algebra associated with a Brauer graph G and $\mathrm{gr}(A)$ the associated graded algebra of A. Then the following statements are equivalent.*

(1) *A is isomorphic to $\mathrm{gr}(A)$ as algebras.*
(2) *The vertices in the Brauer graph G have the same graded degree.*
(3) *$\mathbb{W}$ (resp. $\mathbb{P}$) is an empty set.*

3. ⋆-Condition, unbalanced edge pair and main result

Throughout this section, we assume that $A = kQ/I$ is a Brauer graph algebra associated with a Brauer graph $G = (V(G), E(G), m)$ and that $\mathrm{gr}(A) = kQ/I'$ is its associated graded algebra. Moreover, let $\overline{A} = kQ/I_1$ and $\overline{\mathrm{gr}(A)} = kQ/I_2$ be the associated string algebras in (2.1) and (2.2), respectively. Note that by definition, A and $\overline{A}$ (resp. $\mathrm{gr}(A)$ and $\overline{\mathrm{gr}(A)}$) have the same representation type. Note also that by Proposition 2.8, for $n \geq 1$, $\mathrm{gr}(A)$ is n-domestic if and only if the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is n. In this section, we will define useful notions and state our main results on the infinite representation type of $\mathrm{gr}(A)$.

3.1. ⋆-Condition in a Brauer graph

Definition 3.1 *(Compare with [10, Definition 3.7]).* Let u, v be two distinct vertices in a Brauer graph G.

(1) A walk from u to v is a sequence $[v_1, a_1, v_2, \ldots, v_{k-1}, a_{k-1}, v_k]$ of vertices and edges, where $v_1 = u$, $v_k = v$, a_i is an edge incident to the vertices v_i and v_{i+1} for each $1 \leq i \leq k-1$, and all vertices (and hence all edges) are pairwise distinct. We often simply write this walk by $[a_1, \ldots, a_{k-1}]$ and call it walk from edge a_1 to edge a_{k-1}. In particular, when G is a tree, the walk from u to v is unique.
(2) The length of a walk from u to v is defined to be the number of edges in this walk and will be denoted by $d_G(u, v)$ whenever the context is clear.
(3) We say that a walk $[v_1, a_1, v_2, \ldots, v_{k-1}, a_{k-1}, v_k]$ is degree decreasing if $\mathrm{grd}(v_1) \geq \mathrm{grd}(v_2) \geq \cdots \geq \mathrm{grd}(v_k)$.

Remark 3.2. The definition of a walk in Brauer graph is different from the definition of a walk in graph theory, actually, any walk in Brauer graph is identified with a path in graph theory.

Remark 3.3. According to [10], $\mathrm{gr}(A)$ is of finite representation type if and only if G is a Brauer tree with an exceptional vertex v_0 of multiplicity m_0 such that any walk starting from a specified vertex v_h is degree decreasing, where v_h is defined to be v_0 when $m_0 > 1$ or one of the vertices with maximal graded degree when $m_0 = 1$.

In order to generalize our description from finite representation type to infinite representation type, we introduce the following condition on any Brauer graph.

Definition 3.4. Let G be a Brauer graph and $v_S \overset{i}{\text{---}} v_L$ an unbalanced edge in G. We say that G satisfies ⋆-condition with respect to $v_S \overset{i}{\text{---}} v_L$ if the following three conditions hold:

(1) $G_{i,S} \neq G_{i,L}$, that is, the two subgraphs are not the same subgraph.
(2) $G_{i,S}$ is a tree with $m(v) = 1$ for all $v \in V(G_{i,S})$.
(3) The unique walk from v_S to any vertex v in $G_{i,S}$ is degree decreasing.

Remark 3.5.

(1) $G_{i,L} = G_{i,S}$ for an unbalanced edge $v_S \overset{i}{\text{---}} v_L$ in a Brauer graph G if and only if i is an edge in some cycle of G if and only if there is another walk from v_L to v_S different from $[i]$.
(2) By [10, Theorem 4.5], we can formulate the finite representation type using ⋆-condition as follows: $\mathrm{gr}(A)$ is of finite representation type if and only if G is a Brauer tree which satisfies ⋆-condition with respect to any unbalanced edge in G.

Definition 3.6 *(Compare with [10, Definition 3.9]).* Let $c_n \ldots c_1$ be a string in $\overline{\mathrm{gr}(A)}$. We say that $c_n \ldots c_1$ is a simple string in $\overline{\mathrm{gr}(A)}$ from $s(c_1)$ to $t(c_n)$ if all $s(c_k)$ are pairwise distinct and $t(c_n)$ is different from $s(c_k)$ for each $1 \leq k \leq n$.

Remark 3.7. Similarly as in the proof of [10, Lemma 3.8], for any walk of length ≥ 2 in G we can get exactly two simple strings in $\overline{\mathrm{gr}(A)}$ corresponding to this walk.

We now generalize some results for Brauer tree algebras in [10] to Brauer graph algebras.

Lemma 3.8 *(Compare with [10, Lemma 3.10]). Let $\overline{\mathrm{gr}(A)} = kQ/I_2$ and $C = c_n \dots c_2 c_1$ a string in $\overline{\mathrm{gr}(A)}$, where $s(C) = t(C) = i$. We denote by $u \overset{i}{—} v$ the corresponding edge in G. If c_1 or c_1^{-1} lies in a special cycle at v and $G_{i,v} \neq G_{i,u}$, then C has a substring $C' = c_s \dots c_2 c_1$ satisfying $s(C') = t(C') = i$ such that the edges corresponding to vertices $s(c_k)$ $(2 \leq k \leq s)$ lie in $G_{i,v}$. Moreover, if $G_{i,v}$ is a tree, then C' has a substring C_1 such that $s(C_1) = t(C_1)$ and that C_1 or ${C_1}^{-1}$ is a directed string.*

Proof. The first result is an obvious consequence of $G_{i,v} \neq G_{i,u}$. The proof of the second result is identical to [10, Lemma 3.10]. □

Lemma 3.9 *(Compare with [10, Lemma 4.3]). Let G be a Brauer graph and $\overline{\mathrm{gr}(A)} = kQ/I_2$. Suppose that $C = c_n \dots c_l \dots c_1$ is a string in $\overline{\mathrm{gr}(A)}$ satisfying $l < n$ and that $c_l \dots c_1$ or $c_1^{-1} \dots c_l^{-1}$ is an element of $\mathbb{P}$, where $s(c_1) = t(c_n) = i$. We denote by $v_S \overset{i}{—} v_L$ the corresponding unbalanced edge in G. If $G_{i,S} \neq G_{i,L}$ and $G_{i,S}$ is a tree, then at least one of the following holds.*

(1) *There is a vertex v with $m(v) \geq 2$ in $G_{i,S}$.*
(2) *There are some adjacent vertices v, w in $G_{i,S}$, such that $d_G(v, v_S)+1 = d_G(w, v_S)$ and $\mathrm{grd}(v) < \mathrm{grd}(w)$.*

In other words, G does not satisfy $\star$-condition with respect to $v_S \overset{i}{—} v_L$.

Proof. Noting that [10, Lemma 3.4], [10, Lemma 4.1] and Lemma 3.8, we have an identical proof to [10, Lemma 4.3]. □

The first statement in the following result generalizes [10, Lemma 3.5], and the second one generalizes [10, Lemma 5.1] and [10, Proposition 5.4], both are stated in the Brauer tree case in [10].

Lemma 3.10.

(1) *Let E be a set consisting of some unbalanced edges in G, and for each $i \in E$, let r_i be the element in $\mathbb{P}$ corresponding to the unbalanced edge i, where $\mathbb{P}$ is defined in (2.5). If C is a string in $\overline{\mathrm{gr}(A)}$ and C is not a string in $\overline{\mathrm{gr}(A)}/(\sum_{i \in E} kr_i)$, then, there exists $i \in E$ such that C or C^{-1} has a substring r_i. In particular, if C is a string in $\overline{\mathrm{gr}(A)}$ and C is not a string in $\overline{A}$, then C or C^{-1} has a substring lying in the set $\mathbb{P}$.*
(2) *Suppose that $C = c_n \dots c_l \dots c_1$ is a string in $\overline{\mathrm{gr}(A)}$ satisfying $l < n$ and that $c_l \dots c_1$ or $c_1^{-1} \dots c_l^{-1}$ is an element of $\mathbb{P}$, where $s(c_1) = t(c_l) = i$ and $\mathbb{P}$ is defined in (2.5). Denote by $v_S \overset{i}{—} v_L$ the corresponding unbalanced edge in G. If G satisfies $\star$-condition with respect to $v_S \overset{i}{—} v_L$, then*
    (2.1) *$c_n \dots c_{l+1}$ is a simple substring of C such that $t(c_k)$ is in $G_{i,S}$ for each $l+1 \leq k \leq n$ and in particular the string C is not a band in $\overline{\mathrm{gr}(A)}$;*
    (2.2) *the number of strings C in $\mathrm{St}(\overline{\mathrm{gr}(A)})$ which contain a substring r_i or r_i^{-1} is equal to $(n_i + 1)^2$, where n_i is the number of edges in $G_{i,S}$.*
(3) *Let E be a set consisting of some unbalanced edges in G, and for each $i \in E$, let r_i be the element in $\mathbb{P}$ corresponding to the unbalanced edge i. If G satisfies $\star$-condition with respect to any unbalanced edge in E, then $\overline{\mathrm{gr}(A)}$ and $\overline{\mathrm{gr}(A)}/(\sum_{i \in E} kr_i)$ have the same representation type. In particular, if E is the set of all unbalanced edges in G and G satisfies $\star$-condition with respect to any unbalanced edge, then, by the relationship $\overline{A} \cong \overline{\mathrm{gr}(A)}/(\sum_{i \in E} kr_i)$, $\overline{\mathrm{gr}(A)}$ and $\overline{A}$ have the same representation type.*

Proof. (1) Since the k-vector space $\sum_{i \in E} kr_i$ is an ideal of $\overline{\mathrm{gr}(A)}$, the conclusion is clear.

(2.1) To show that the edge corresponding to $t(c_k)$ is in $G_{i,S}$ for $l+1 \le k \le n$, it suffices to prove that $t(c_k) \neq i$ for $l+1 \le k \le n$. Suppose on the contrary that there exists $l+1 \le m \le n$ such that $t(c_m) = i$ and $t(c_k) \neq i$ for $l+1 \le k \le m-1$. Since G satisfies $\star$-condition with respect to $v_S \overset{i}{\text{——}} v_L$, we have that $G_{i,S} \neq G_{i,L}$ and $G_{i,S}$ is a tree. For the substring $c_m \ldots c_l \ldots c_1$, where $s(c_1) = t(c_m) = i$, the string $c_m \ldots c_l \ldots c_1$ satisfies the conditions of Lemma 3.9, which then leads to a contradiction.

Next we show that $c_n \ldots c_{l+1}$ is a simple string. It suffices to show that all $t(c_k)$ are pairwise distinct for $l \le k \le n$. Suppose that there exist k and t satisfying $l \le t < k \le n$ such that $t(c_k) = t(c_t) = s(c_{t+1})$ and that $t(c_m)$ is different from $t(c_s)$ for each $l \le m < k$ and $l \le s < m$. Repeating the similar proof as above, we still get a contradiction.

(2.2) Since G satisfies $\star$-condition with respect to $v_S \overset{i}{\text{——}} v_L$, this proof is identical to the proof of [10, Proposition 5.4].

(3) Since G satisfies $\star$-condition with respect to any unbalanced edge in E, (2.1) shows that the band modules over the two algebras $\overline{\mathrm{gr}(A)}$ and $\overline{\mathrm{gr}(A)}/(\sum_{i\in E} kr_i)$ are the same, and (2.2) shows that the number of string $\overline{\mathrm{gr}(A)}$-modules is equal to the number of string $\overline{\mathrm{gr}(A)}/(\sum_{i\in E} kr_i)$-modules plus $\sum_{i\in E}(n_i+1)^2$, it follows that $\overline{\mathrm{gr}(A)}$ and $\overline{\mathrm{gr}(A)}/(\sum_{i\in E} kr_i)$ have the same representation type. □

Proposition 3.11. *Let G be a Brauer graph which is a tree with $m(v) = 2$ for exactly two vertices $v = w_0, w_1$ in $V(G)$ and $m(v) = 1$ for all $v \neq w_0, w_1$. Then the following two conditions are equivalent:*

(1) *G satisfies $\star$-condition with respect to any unbalanced edge $v_S \overset{i}{\text{——}} v_L$;*
(2) *w_0 and w_1 are in $G_{i,L}$ for any unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G.*

Moreover, if G satisfies $\star$-condition with respect to any unbalanced edge, then $\mathrm{gr}(A)$ *is domestic. In particular,* $\mathrm{gr}(A)$ *is* 1*-domestic.*

Proof. Assume that w_0 and w_1 are in $G_{i,L}$ for any unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G. Since G is a tree with $m(v) = 2$ for exactly $v = w_0, w_1$ and w_0 and w_1 are in $G_{i,L}$ for $v_S \overset{i}{\text{——}} v_L$, $G_{i,S} \neq G_{i,L}$ and $G_{i,S}$ is a tree with $m \equiv 1$. Then the conditions (1) and (2) of $\star$-condition hold. We next show that the condition (3) of $\star$-condition holds.

We suppose, on the contrary that, there exists a vertex w in $G_{i,S}$ for some unbalanced edge $v_S \overset{i}{\text{——}} v_L$ such that the walk $[v_1, a_1, v_2, \ldots, v_{k-1}, a_{k-1}, v_k]$ from v_S to w is not degree decreasing, where $v_1 = v_S$ and $v_k = w$. In other words, there exists an unbalanced edge $v_j \overset{a_j}{\text{——}} v_{j+1}$ with $\mathrm{grd}(v_j) < \mathrm{grd}(v_{j+1})$ for some $1 \le j \le k-1$. Since $d_G(v_j, v_L) + 1 = d_G(v_{j+1}, v_L)$ and w_0 and w_1 are in $G_{i,L}$, $G_{i,L} \subseteq G_{a_j,S}$ and w_0 and w_1 are in $G_{a_j,S}$. It contradicts the condition that w_0 and w_1 are in $G_{a_j,L}$.

Conversely, assume that G satisfies $\star$-condition with respect to any unbalanced edge $v_S \overset{i}{\text{——}} v_L$. Suppose that there is some unbalanced edge $v_S \overset{i}{\text{——}} v_L$ such that w_0 or w_1 is in $G_{i,S}$. It clearly contradicts the condition (2) of $\star$-condition.

Now assume that G satisfies $\star$-condition with respect to any unbalanced edge. Then, by Lemma 3.10, $\overline{\mathrm{gr}(A)}$ and $\overline{A}$ have the same representation type. It follows from Theorem 2.9 that $\overline{\mathrm{gr}(A)}$ is 1-domestic. Hence $\mathrm{gr}(A)$ is 1-domestic. □

Proposition 3.12. *Let G be a Brauer graph with a unique cycle and $m(v) = 1$ for all $v \in V(G)$. Then the following two conditions are equivalent:*

(1) *G satisfies $\star$-condition with respect to any unbalanced edge $v_S \overset{i}{\text{——}} v_L$;*
(2) *all edges in the unique cycle are not unbalanced edges and the unique cycle is in $G_{i,L}$ for any unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G.*

Moreover, if G satisfies $\star$-condition with respect to any unbalanced edge, then $\mathrm{gr}(A)$ *is domestic. In particular, if the unique cycle is of odd length (resp. even length), then* $\mathrm{gr}(A)$ *is* 1*-domestic (resp.* 2*-domestic).*

Proof. Assume that all edges in the unique cycle are not unbalanced edges and the unique cycle is in $G_{i,L}$ for any unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G. Since $m(v) = 1$ for all $v \in V(G)$, $m(v) = 1$ for all $v \in V(G_{i,S})$ for $v_S \overset{i}{\text{——}} v_L$. Since all edges in the unique cycle are not unbalanced edges and the unique cycle is in $G_{i,L}$ for $v_S \overset{i}{\text{——}} v_L$, $G_{i,S} \neq G_{i,L}$ and $G_{i,S}$ is a tree. Then the conditions (1) and (2) of $\star$-condition hold. The condition (3) of $\star$-condition holds by using a similar approach to the proof of Proposition 3.11.

Conversely, assume that G satisfies $\star$-condition with respect to any unbalanced edge $v_S \overset{i}{\text{——}} v_L$. If there is some edge in the unique cycle is an unbalanced edge, then it contradicts the condition (1) of $\star$-condition. If there is some unbalanced edge $v_S \overset{i}{\text{——}} v_L$ such that the unique cycle is in $G_{i,S}$, then it contradicts the condition (2) of $\star$-condition.

Now assume that G satisfies $\star$-condition with respect to any unbalanced edge. Then, by Lemma 3.10, $\overline{\mathrm{gr}(A)}$ and $\overline{A}$ have the same representation type. It follows from Theorem 2.9 that if the unique cycle in G is of odd length (resp. even length), then $\mathrm{gr}(A)$ is 1-domestic (resp. 2-domestic). □

The next result deals with a case where the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is infinite.

Lemma 3.13. *Suppose that there are two distinct bands $b_1 = c_m \dots c_{l+1}c_l \dots c_1$ and $b_2 = c'_{m'} \dots c'_{l+1}c_l \dots c_1$ in $\overline{\mathrm{gr}(A)}$, where $s(c_1) = t(c_l)$, $c_l \dots c_1$ is a directed substring, $c_{l+1} = c'_{l+1}$ is an inverse arrow, and $s(c_i) \neq s(c_1)$ (resp. $s(c'_i) \neq s(c_1)$) for $l + 1 < i \leq m$ (resp. $l + 1 < i \leq m'$). Then the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is infinite and $\mathrm{gr}(A)$ is not of polynomial growth.*

Proof. From the assumption of the two bands b_1 and b_2, we have that all powers of b_2b_1 are strings in $\overline{\mathrm{gr}(A)}$. Moreover, since b_1 and b_2 are distinct and $s(c_i) \neq s(c_1)$ (resp. $s(c'_i) \neq s(c_1)$) for $l + 1 < i \leq m$ (resp. $l + 1 < i \leq m'$), b_2b_1 is not a power of a string of smaller length. Then b_2b_1 is also a band. Similarly, for any positive integer k, $b_2^k b_1$ is a band. Then the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is infinite.

In order to prove that $\mathrm{gr}(A)$ is not of polynomial growth, we just need to prove that $\overline{\mathrm{gr}(A)}$ is not of polynomial growth. The band b_1 (resp. b_2) has length m (resp. m'). When $m = m'$, there are pairwise distinct elements (indexed by $(k_1, k_2, \cdots, k_n)$)

$$(*) \quad b_2^{k_n-1} b_1^{k_n} b_2^{k_{n-1}+1} b_1^{k_{n-1}} b_2^{k_{n-2}} b_1^{k_{n-2}} \dots b_2^{k_1} b_1^{k_1}$$

in $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$, where n and k_n are positive integers greater than 1, and k_i's $(1 \leqslant i \leqslant n-1)$ are positive integers such that $\sum_{i=1}^{n} k_i = tm$ for some positive integer t. Note that the above band $(*)$ has length $2tm^2$. Fix $t >> 0$ and consider the indecomposable band modules corresponding to the bands $(*)$ and with dimension d (where $d = 2tm^2$), we have that the number of this kind of indecomposable band modules is $\sum_{n=2}^{tm-1} \binom{tm-2}{n-1} = 2^{tm-2} - 1$. Therefore $\mu_{\overline{\mathrm{gr}(A)}}(d) \geq 2^{tm-2} - 1$ and there is no positive integer s such that $\mu_{\overline{\mathrm{gr}(A)}}(d) \leq d^s$ for all $d \geq 2$. Hence $\mathrm{gr}(A)$ is not of polynomial growth.

When $m \neq m'$, without loss of generality, we can assume $m' < m$. Similarly, there are pairwise distinct elements (indexed by $(k_1, k_2, \cdots, k_n)$)

$$(**) \quad b_2^{k_n + t(m-m')} b_1^{k_n} b_2^{k_{n-1}} b_1^{k_{n-1}} \dots b_2^{k_1} b_1^{k_1}$$

in $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$, where n and k_i's $(1 \leqslant i \leqslant n-1)$ are positive integers such that $\sum_{i=1}^{n} k_i = tm'$. Note that the above band $(**)$ has length $2tmm'$. Fix $t >> 0$ and consider the indecomposable band modules corresponding to the bands $(**)$ and with dimension d (where $d = 2tmm'$), we have that the number of this kind of indecomposable band modules is $\sum_{n=1}^{tm'} \binom{tm'-1}{n-1} = 2^{tm'-1}$. Therefore $\mu_{\overline{\mathrm{gr}(A)}}(d) \geq 2^{tm'-1}$ and there is no positive integer s such that $\mu_{\overline{\mathrm{gr}(A)}}(d) \leq d^s$ for all $d \geq 2$. Hence $\mathrm{gr}(A)$ is not of polynomial growth. □

3.2. Unbalanced edge pair in a Brauer tree

In order to describe the domestic $\overline{\mathrm{gr}(A)}$ when G is a Brauer tree, we introduce the notion of unbalanced edge pair.

Definition 3.14. Let G be a Brauer tree with an exceptional vertex v_0 of multiplicity m_0.

(1) We call the set $\{i,j\}$ an unbalanced edge pair if j is an edge in $G_{i,S}$ and $d_G(v_S^{(j)}, v_S^{(i)})+1 = d_G(v_L^{(j)}, v_S^{(i)})$, where $v_S^{(i)} \overset{i}{\text{——}} v_L^{(i)}$ and $v_S^{(j)} \overset{j}{\text{——}} v_L^{(j)}$ are two unbalanced edges in G.
(2) We define κ_0 to be the number of unbalanced edges $v_S \overset{i}{\text{——}} v_L$ in G such that the exceptional vertex v_0 is a vertex in $G_{i,S}$.
(3) We define κ_1 to be the number of unbalanced edge pairs in G.

Remark 3.15.

(1) $\kappa_1 = 0$ if and only if the unique walk from v_S to any vertex in $G_{i,S}$ is degree decreasing for any unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G.
(2) By [10, Theorem 4.5], $\mathrm{gr}(A)$ is of finite representation type if and only if G is a Brauer tree such that $\kappa_0(m_0-1)+\kappa_1 = 0$.

We have the following observations about κ_0 and κ_1.

Lemma 3.16. *If $\kappa_1 \neq 0$, then there are two unbalanced edges $v_S^{(i)} \overset{i}{\text{——}} v_L^{(i)}$ and $v_S^{(j)} \overset{j}{\text{——}} v_L^{(j)}$ in G such that the exceptional vertex v_0 is a vertex in $G_{i,S}$ and $\{i,j\}$ is an unbalanced edge pair. In particular, if $\kappa_1 \neq 0$, then $\kappa_0 \neq 0$.*

Proof. Since $\kappa_1 \neq 0$, there are two unbalanced edges $v_S^{(i)} \overset{i}{\text{——}} v_L^{(i)}$ and $v_S^{(j)} \overset{j}{\text{——}} v_L^{(j)}$ in G such that $\{i,j\}$ is an unbalanced edge pair. Without loss of generality, we assume that v_0 is a vertex in $G_{i,L}$. Since $G\setminus i = G_{i,S}\bigcup G_{i,L}$ and $\{i,j\}$ is an unbalanced edge pair, $G_{i,L}\subseteq G_{j,S}$ and v_0 is a vertex in $G_{j,S}$. We get our desired result. $\square$

Lemma 3.17. *Let G be a Brauer tree with an exceptional vertex v_0 of multiplicity m_0. Then $\kappa_1 \geq 2$ if and only if there are three unbalanced edges $v_S^{(i)} \overset{i}{\text{——}} v_L^{(i)}$, $v_S^{(j)} \overset{j}{\text{——}} v_L^{(j)}$ and $v_S^{(k)} \overset{k}{\text{——}} v_L^{(k)}$ in G such that $\{i,j\}$ and $\{i,k\}$ are unbalanced edge pairs.*

Proof. "$\Longleftarrow$" It is obvious to get $\kappa_1 \geq 2$ if $\{i,j\}$ and $\{i,k\}$ are unbalanced edge pairs.

"$\Longrightarrow$" Since $\kappa_1 \geq 2$, there are at least two unbalanced edge pairs. Without loss of generality, we assume that $\{i,j\}$ and $\{k,l\}$ are two unbalanced edge pairs, where the unbalanced edges $v_S^{(i)} \overset{i}{\text{——}} v_L^{(i)}$, $v_S^{(j)} \overset{j}{\text{——}} v_L^{(j)}$, $v_S^{(k)} \overset{k}{\text{——}} v_L^{(k)}$ and $v_S^{(l)} \overset{l}{\text{——}} v_L^{(l)}$ are pairwise distinct and k is an edge in $G_{i,S}$. There are two cases to be considered.

Case 1. If $d_G(v_S^{(k)}, v_S^{(i)})+1 = d_G(v_L^{(k)}, v_S^{(i)})$, then $\{i,k\}$ is an unbalanced edge pair. Moreover, $\{i,j\}$ is also an unbalanced edge pair. We have that $\{i,j\}$ and $\{i,k\}$ are two unbalanced edge pairs.

Case 2. If $d_G(v_S^{(k)}, v_S^{(i)})-1 = d_G(v_L^{(k)}, v_S^{(i)})$, since the unbalanced edge l is an edge in $G_{k,S}$ with $d_G(v_S^{(l)}, v_S^{(k)})+1 = d_G(v_L^{(l)}, v_S^{(k)})$ and $G_{k,S}\subseteq G_{i,S}$, then l is an edge in $G_{i,S}$ with $d_G(v_S^{(l)}, v_S^{(i)})+1 = d_G(v_L^{(l)}, v_S^{(i)})$ and therefore $\{i,l\}$ is an unbalanced edge pair. Moreover, $\{i,j\}$ is also an unbalanced edge pair. We have that $\{i,j\}$ and $\{i,l\}$ are two unbalanced edge pairs. $\square$

3.3. Main result and consequences

We now state our main result, which characterizes when $\mathrm{gr}(A)$ is domestic.

Theorem 3.18. *Let A be the Brauer graph algebra associated with a Brauer graph $G = (V(G), E(G), m)$ and $\mathrm{gr}(A)$ the graded algebra associated with the radical filtration of A, where $V(G)$ is the vertex set, $E(G)$ is the edge set and m is the multiplicity function of G. Let κ_0 and κ_1 be defined as in Definition 3.14. Then the following three statements are equivalent.*

(a) $\mathrm{gr}(A)$ *is of polynomial growth.*
(b) $\mathrm{gr}(A)$ *is domestic.*
(c) *The cardinality of* $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ *is finite, where* $\overline{\mathrm{gr}(A)}$ *is defined in* (2.2).

Furthermore, for domestic type, we have the following.

$(b1)$ $\mathrm{gr}(A)$ *is* 1*-domestic if and only if one of the following holds.*
- (1) *G is a Brauer tree with an exceptional vertex v_0 of multiplicity m_0 such that $\kappa_0(m_0 - 1) + \kappa_1 = 1$.*
- (2) *G is a tree and there exist two distinct vertices w_0, w_1, such that the following conditions hold:*
  - (2.1) *$m(w_0) = m(w_1) = 2$ and $m(v) = 1$ for $v \neq w_0, w_1$.*
  - (2.2) $\mathrm{grd}(w_0) = \mathrm{grd}(w_1)$.
  - (2.3) *Any walk from w_0 (or from w_1) is degree decreasing.*
- (3) *G is a graph with a unique cycle of odd length and $m(v) = 1$ for all $v \in V(G)$, and satisfies the following conditions hold:*
  - (3.1) *$\mathrm{grd}(u) = \mathrm{grd}(v)$ for any two vertices u and v in the unique cycle.*
  - (3.2) *Any walk from any vertex in the unique cycle is degree decreasing.*

$(b2)$ $\mathrm{gr}(A)$ *is* 2*-domestic if and only if G satisfies the following conditions.*
- (1) *G is a graph with a unique cycle of even length and $m(v) = 1$ for all $v \in V(G)$.*
- (2) *$\mathrm{grd}(u) = \mathrm{grd}(v)$ for any two vertices u and v in the unique cycle.*
- (3) *Any walk from any vertex in the unique cycle is degree decreasing.*

$(b3)$ $\mathrm{gr}(A)$ *is not n-domestic for $n \geq 3$.*

Proof. Since A and $\overline{A}$ (resp. $\mathrm{gr}(A)$ and $\overline{\mathrm{gr}(A)}$) have the same representation type, and since $\overline{A}$ is a quotient of the algebra $\overline{\mathrm{gr}(A)}$, if the Brauer graph algebra A is nondomestic (resp. A is not of polynomial growth), then $\mathrm{gr}(A)$ is nondomestic (resp. $\mathrm{gr}(A)$ is not of polynomial growth). By Theorem 2.9, in order to describe when $\mathrm{gr}(A)$ is domestic or is of polynomial growth, it suffices to study $\overline{\mathrm{gr}(A)}$ in the cases $(a), (b), (c)$ in Theorem 2.9. The descriptions in these cases are given in Proposition 4.10, Proposition 5.4 and Proposition 6.6, respectively. □

The main result gives the following consequence.

Corollary 3.19. *Let A be a Brauer tree algebra and $\mathrm{gr}(A)$ the associated graded algebra of A. If one of the following is satisfied:*

(1) *$\kappa_1 > 1$,*
(2) *$\kappa_1 = 1$ and $m_0 > 1$,*

then $\mathrm{gr}(A)$ is not of polynomial growth.

Comparing with the results in [10], we would also like to give the following remarks on the relationships between the Auslander-Reiten quivers of gr(A) and A.

Remark 3.20.

(1) Similar as the discussion in [10, Section 5], for domestic gr(A) except the Brauer tree case, based on Lemma 3.10, Proposition 3.11, Proposition 3.12, we can prove that, the Auslander-Reiten quiver of $\overline{A}$ is obtained from the Auslander-Reiten quiver of $\overline{\mathrm{gr}(A)}$ by removing several diamonds.

(2) When G is a Brauer tree and gr(A) is domestic, the situation is more complicated. In this case we have the following conjecture on the Auslander-Reiten quiver Γ of $\overline{\mathrm{gr}(A)}$:
   (2.1) Γ consists of components of the form $\mathbb{Z}\tilde{A}_{p,q}$ and components of the form $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ (both components are up to deleting some diamonds). Moreover, when $m_0 = 1$, Γ has a component $\mathbb{Z}\tilde{A}_{p,q}$ satisfying $p + q = n_i + n_j + 2$ with $\{i, j\}$ the unique unbalanced edge pair in G; when $m_0 = 2$, Γ has a component $\mathbb{Z}\tilde{A}_{n_i+1,|E(G)|}$ with i the unique unbalanced edge in G such that the exceptional vertex v_0 is in $G_{i,S}$.

(3) We note that in the picture of [10, Remark 5.14], the obtained part W in the Auslander-Reiten quiver of $\overline{\mathrm{gr}(A)}$ may be different from the beginning wing W, since the obtained part may contain new inserted diamonds.

4. The case that G is a Brauer tree

In this section, we describe when $\mathrm{gr}(A) = kQ/I'$ is domestic in the case when $G = (V(G), E(G), m)$ is a Brauer tree with an exceptional vertex v_0 of multiplicity m_0, where $V(G)$ is the vertex set, $E(G)$ is the edge set and m is the multiplicity function of G. Let κ_0 and κ_1 be defined in Definition 3.14.

Recall that since the number of string modules of a given dimension is finite, it suffices to consider band modules when we consider the representation type of a representation-infinite string algebra. The following lemma is useful when we consider two related representation-infinite string algebras.

Lemma 4.1. *Let $\Lambda = kQ/I$ and $\Gamma = \Lambda/J$ be two representation-infinite string algebras, where J is an ideal of Λ with $\mathrm{rad}^m(\Lambda) \subseteq J \subseteq \mathrm{rad}^2(\Lambda)$ for some $m \geq 2$. Suppose that for any indecomposable Λ-module M satisfying $JM \neq 0$, M is a string Λ-module. Then Γ is of polynomial growth (resp. domestic) if and only if Λ is of polynomial growth (resp. domestic).*

Proof. Since the algebra Γ is a quotient of the algebra Λ, we have that any band Γ-module can be considered as a band Λ-module. Moreover, from the assumption that M is a string Λ-module for any indecomposable Λ-module M satisfying $JM \neq 0$, it follows that any band Λ-module is also a band Γ-module. Hence there is a one to one correspondence between band Λ-modules and band Γ-modules. Combining the remark before this lemma, we get the desired result. □

Remark 4.2. We have used a special case of the above lemma in Lemma 3.10 (3), where $\Lambda = \overline{\mathrm{gr}(A)}$ and $\Gamma = \overline{A}$.

The strategy when we prove that the string algebra $\overline{\mathrm{gr}(A)}$ is not of polynomial growth. Before giving the following result we would like to state two main methods used in this paper when we want to show $\overline{\mathrm{gr}(A)}$ is not of polynomial growth. The first method is to construct two bands b_1 and b_2 in $\overline{\mathrm{gr}(A)}$ and then use Lemma 3.13 or use similar method as in Lemma 3.13. The second method is to construct a representation-infinite quotient algebra C of $\overline{\mathrm{gr}(A)}$ such that C is also a quotient algebra of some Brauer graph algebra B which is representation-infinite and not of polynomial growth:

$$\overline{\mathrm{gr}(A)} \twoheadrightarrow C \twoheadleftarrow B,$$

and then use Lemma 4.1. In some situations (for example, in next proposition), both methods work well.

Proposition 4.3. *Let $\overline{\mathrm{gr}(A)} = kQ/I_2$ be defined in (2.2) and G the associated Brauer tree with an exceptional vertex v_0 of multiplicity m_0. If $m_0 \geq 3$ and $\kappa_0 \neq 0$, then $\overline{\mathrm{gr}(A)}$ and $\mathrm{gr}(A)$ are not of polynomial growth.*

Proof. It is enough to show that $\overline{\mathrm{gr}(A)}$ is not of polynomial growth. We will prove this using the second method mentioned above. The proof is divided into three steps.

Step 1. We prove that there is a quotient algebra C (which is a representation-infinite string algebra) of $\overline{\mathrm{gr}(A)}$.

Since $\kappa_0 \neq 0$, by definition there is an unbalanced edge $v_S \overset{i}{\text{——}} v_L$ such that the exceptional vertex v_0 is in $G_{i,S}$. Let $i_1 < i_2 < \cdots < i_n < i_1$ be the cyclic ordering at v_L, where $i_1 = i$ and $n = \mathrm{grd}(v_L) = \mathrm{val}(v_L)$. Note that $n > 2$. Let $E_1 = \{i_1, i_2, \ldots, i_n\}$ and $E_2 = E(G_{i,S}) \cup E_1$, where the edge i_k is incident to v_L and v'_k for any $2 \leq k \leq n$, and $E(G_{i,S})$ is the edge set of $G_{i,S}$. We denote by E_3 the set of all unbalanced edges in $G_{i,S}$ and by r_j the element in $\mathbb{P}$ corresponding to the unbalanced edge j in E_3. We have algebra epimorphisms as follows.

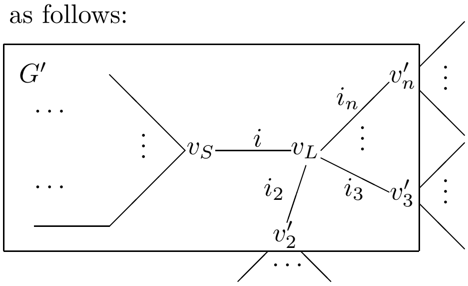
$$\mathrm{gr}(A) \twoheadrightarrow \overline{\mathrm{gr}(A)} \twoheadrightarrow \overline{\mathrm{gr}(A)}/(\overline{\mathrm{gr}(A)}e\overline{\mathrm{gr}(A)} \oplus \sum_{j \in E_3} kr_j),$$

where $e = \sum_{i \in E(G) \setminus E_2} e_i$ and e_i is the primitive idempotent in Q corresponding to the edge i in $E(G) \setminus E_2$.

Let C be the above algebra $\overline{\mathrm{gr}(A)}/(\overline{\mathrm{gr}(A)}e\overline{\mathrm{gr}(A)} \oplus \sum_{j \in E_3} kr_j)$. Then $C = kQ'/I_C$, where I_C is an admissible ideal in kQ' and Q' is a subquiver of Q by removing all vertices corresponding to the edges in $E(G) \setminus E_2$ and all related arrows. Note that C is a string algebra and is representation-infinite.

Step 2. We prove that C is a quotient algebra of some Brauer graph algebra B which is representation-infinite and not of polynomial growth.

We next construct a related Brauer graph G'. Let $G' = (V(G_{i,S}) \cup \{v_L, v'_2, v'_3, \ldots, v'_n\}, E_2, m)$, where $V(G_{i,S})$ is the vertex set of $G_{i,S}$, $m(v_0) = m_0$, $m(v_L) = 2$ and $m(v) = 1$ for the other vertices v. Note that the vertex v_S is not truncated in G' since the edge i is still unbalanced in G'. We may visualise the underlying graph of G' from G as follows:



Let B be the Brauer graph algebra associated with the new Brauer graph G' and $\overline{B} = kQ'/I_{B,1}$ the corresponding string algebra defined in (2.1). By the construction of the Brauer graph G' and Theorem 2.9, we have that B is representation-infinite and not of polynomial growth. Therefore, $\overline{B}$ is not of polynomial growth.

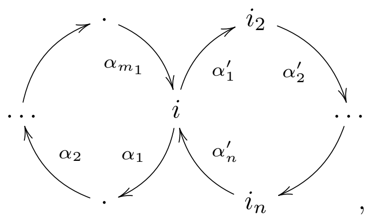
Now let $E_4 = \{i_k | \mathrm{grd}(v_L) \leq \mathrm{grd}(v'_k), 1 < \mathrm{val}(v'_k), 2 \leq k \leq n\}$ (which is also defined in the original Brauer graph G), where the edge i_k is incident to v_L and v'_k in G for any $2 \leq k \leq n$. We can get the following algebra isomorphism from their constructions

$$C \cong \overline{B}/(\bigoplus_{i_k \in E_1 \setminus E_4} \mathrm{rad}^{n+1}(P_{i_k}) \oplus \bigoplus_{i_k \in E_4} \mathrm{rad}^{n}(P_{i_k})),$$

where P_{i_k} is the projective cover of the simple $\overline{B}$-module S_{i_k} corresponding to the edge i_k in G' for each $1 \le k \le n$ and where $J := \bigoplus_{i_k \in E_1 \setminus E_4} \operatorname{rad}^{n+1}(P_{i_k}) \oplus \bigoplus_{i_k \in E_4} \operatorname{rad}^n(P_{i_k})$ is an ideal of $\overline{B}$. Clearly $J \subseteq \operatorname{rad}^2(\overline{B})$.

Step 3. We prove that C is not of polynomial growth and therefore $\overline{\operatorname{gr}(A)}$ is not of polynomial growth.

Note that there are two arrows starting and ending at vertex i of Q' and there is one arrow starting and ending at vertex i_k of Q' for all $2 \le k \le n$. So Q' contains the following subquiver:



where m_1 denotes the valency of v_S. Moreover, J can be generated by $\alpha'_{k-1} \dots \alpha'_1 \alpha'_n \dots \alpha'_{k+1} \alpha'_k$ ($i_k \in E_4$) and $\alpha'_k \dots \alpha'_1 \alpha'_n \dots \alpha'_{k+1} \alpha'_k$ ($i_k \in E_1 \setminus E_4$). For any band in $\overline{B}$, we have the following claim.

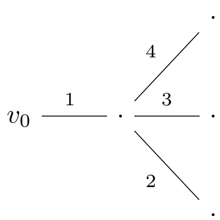
Claim: For any band b in $\overline{B}$, b does not have the substring $\alpha'_{k-1} \dots \alpha'_1 \alpha'_n \dots \alpha'_{k+1} \alpha'_k$ for any i_k in E_4, and b does not have the substring $\alpha'_k \dots \alpha'_1 \alpha'_n \dots \alpha'_{k+1} \alpha'_k$ for any i_k in $E_1 \setminus E_4$ (possibly after rotation or taking inverse of b). That is, any band in $\overline{B}$ gives in fact a band in the quotient algebra C.

If the above claim is true, then, the ideal J satisfies the condition in Lemma 4.1, and therefore C is not of polynomial growth. It follows that $\overline{\operatorname{gr}(A)}$ is not of polynomial growth. This is our desired result.

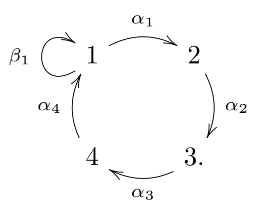
Proof of Claim. Suppose on the contrary that b has the substring $\alpha'_{k-1} \dots \alpha'_1 \alpha'_n \dots \alpha'_{k+1} \alpha'_k$ for some $i_k \neq i_1$ and $b = c_s \dots c_1 \alpha'_{k-1} \dots \alpha'_1 \alpha'_n \dots \alpha'_{k+1} \alpha'_k$. Since $s(b) = t(b)$ and there is only one arrow starting and ending at vertex i_k for $2 \le k \le n$, $s(\alpha'_k) = t(c_s) = i_k$ and $c_s = \alpha'_{k-1}$. We rotate b to $c_{s-1} \dots c_1 \alpha'_{k-1} \dots \alpha'_1 \alpha'_n \dots \alpha'_k \alpha'_{k-1}$. If $i_{k-1} \neq i_1$, then we can repeat the above step and therefore we can assume that b has the substring $\alpha'_1 \alpha'_n \dots \alpha'_2 \alpha'_1$. We may assume that $b = c_s \dots c_1 \alpha'_1 \alpha'_n \dots \alpha'_2 \alpha'_1$. Since $s(\alpha'_1) = t(c_s)$ and there is only one arrow starting and ending at vertex i_k for $2 \le k \le n$, we have that b has the substring $\alpha'_n \dots \alpha'_2 \alpha'_1 \alpha'_n \dots \alpha'_2 \alpha'_1$. It contradicts the fact that $\alpha'_n \dots \alpha'_2 \alpha'_1 \alpha'_n \dots \alpha'_2 \alpha'_1$ is in the ideal $I_{B,1}$ (indeed it is an element of $\operatorname{soc}(B)$). This finishes the proof of our claim. □

We give an example to illustrate the above result.

Example 4.4. Let G be the following Brauer tree with $m_0 = 3$.



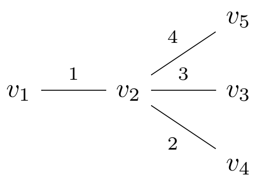
Let $A = kQ/I$ be the Brauer tree algebra associated with G and $\operatorname{gr}(A)$ the associated graded algebra of A. The quiver Q is as follows.



The regular representation of $\mathrm{gr}(A)$ is as follows.

$$\begin{array}{cccccccccc}
 & 1 & & & 2 & & 3 & & 4 \\
1 & & 2 & & 3 & & 4 & & 1 \\
 & & 3 & \oplus & 4 & \oplus & 1 & \oplus & 2 \\
1 & & 4 & & 1 & & 2 & & 3 \\
 & & 1 & & 2 & & 3 & & 4
\end{array}$$

Note that $E_1 = E_2 = E(G) = \{1, 2, 3, 4\}, E_3 = \emptyset$ and $\mathrm{gr}(A) = \overline{\mathrm{gr}(A)} = C$, $b_1 = (\beta_1)^{-1}\alpha_4\alpha_3\alpha_2\alpha_1$ and $b_2 = (\beta_1)^{-1}(\beta_1)^{-1}\alpha_4\alpha_3\alpha_2\alpha_1$ are bands in $\overline{\mathrm{gr}(A)}$. By a method similar to Lemma 3.13, we can show that there are infinitely many bands in $\overline{\mathrm{gr}(A)}$. We have that G' is the following Brauer graph



where $m(v_1) = 3$, $m(v_2) = 2$ and $m(v_3) = m(v_4) = m(v_5) = 1$. The regular representation of the corresponding Brauer graph algebra B is as follows.

$$\begin{array}{cccccccccc}
 & 1 & & & 2 & & 3 & & 4 \\
 & & 2 & & 3 & & 4 & & 1 \\
 & & 3 & & 4 & & 1 & & 2 \\
1 & & 4 & & 1 & & 2 & & 3 \\
 & & 1 & \oplus & 2 & \oplus & 3 & \oplus & 4 \\
 & & 2 & & 3 & & 4 & & 1 \\
1 & & 3 & & 4 & & 1 & & 2 \\
 & & 4 & & 1 & & 2 & & 3 \\
 & 1 & & & 2 & & 3 & & 4
\end{array}$$

Note that $\overline{B} = B/\operatorname{soc}(P_1)$, $E_4 = \emptyset$ and $C \cong \overline{B}/\operatorname{rad}^5(P_1 \oplus P_2 \oplus P_3 \oplus P_4)$, where P_i is the projective cover of the simple $\overline{B}$-module S_i corresponding to the edge i in G'. Since B is not of polynomial growth and $\overline{B}$ is not of polynomial growth, C is not of polynomial growth and therefore $gr(A)$ is not of polynomial growth.

Lemma 4.5. *Let* $\overline{\mathrm{gr}(A)} = kQ/I_2$ *be defined in* (2.2). *If one of the following is satisfied:*

(1) $m_0 = 1$ *and* $\kappa_1 = 1$,
(2) $m_0 = 2$, $\kappa_0 = 1$ *and* $\kappa_1 = 0$,

then $\mathrm{gr}(A)$ *is* 1*-domestic and therefore the cardinality of* $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ *is* 1.

Proof. By Proposition 2.8, we can give a proof by counting the number of bands in $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$. However, we would like give a more conceptual proof using a similar method as in the proof of Proposition 4.3.

If $m_0 = 1$ and $\kappa_1 = 1$, then there are only two unbalanced edges $v_S^{(i)} \overset{i}{\text{---}} v_L^{(i)}$ and $v_S^{(j)} \overset{j}{\text{---}} v_L^{(j)}$ in G such that j is in $G_{i,S}$ and $d_G(v_S^{(j)}, v_S^{(i)}) + 1 = d_G(v_L^{(j)}, v_S^{(i)})$. Let $i_1 < i_2 < \cdots < i_t < i_1$ (resp. $j_1 < j_2 < \cdots < j_{t_1} < j_1$) be the cyclic ordering at $v_L^{(i)}$ (resp. $v_L^{(j)}$), where $i_1 = i$ (resp. $j_1 = j$) and $t = \mathrm{val}(v_L^{(i)})$ (resp. $t_1 = \mathrm{val}(v_L^{(j)})$). Note that $t > 2$ and $t_1 > 2$. Let $E_1 = \{i_1, i_2, \dots, i_t\}$ and $E_2 = \{j_1, j_2, \dots, j_{t_1}\}$, where the edge i_k is incident to $v_L^{(i)}$ and v'_k for any $2 \le k \le t$, and the edge j_k is incident to $v_L^{(j)}$ and w'_k for any

$2 \le k \le t_1$. We denote by E_3 the set of all unbalanced edges different from i and j in G, and by r_l the element in $\mathbb{P}$ corresponding to an unbalanced edge l in E_3. There are algebra epimorphisms as follows.

$$\mathrm{gr}(A) \twoheadrightarrow \overline{\mathrm{gr}(A)} \twoheadrightarrow \overline{\mathrm{gr}(A)}/(\sum_{l\in E_3} kr_l).$$

Since $m_0 = 1$ and $\kappa_1 = 1$, we have that $\mathrm{grd}(u) \ge \mathrm{grd}(v)$ for any edge u——v different from i and j in G satisfying $d_G(u, v_S^{(i)}) + 1 = d_G(v, v_S^{(i)})$, and the unique walk $[v_1, a_1, v_2, \dots, v_{k-1}, a_{k-1}, v_k]$ from $v_S^{(i)}$ to $v_S^{(j)}$ satisfies $\mathrm{grd}(v_1) = \mathrm{grd}(v_2) = \dots = \mathrm{grd}(v_k)$, where $v_1 = v_S^{(i)}$, $v_k = v_S^{(j)}$, and a_i is an edge incident to the vertices v_i and v_{i+1} for each $1 \le i \le k-1$. Then G satisfies $\star$-condition with respect to any unbalanced edge in E_3. Therefore, by Lemma 3.10, $\overline{\mathrm{gr}(A)}$ and $\overline{\mathrm{gr}(A)}/(\sum_{l\in E_3} kr_l)$ have the same representation type. In particular, since $\kappa_0(m_0 - 1) + \kappa_1 = 1 \ne 0$, they are of infinite representation type.

Let $G' = (V(G), E(G), m)$ be a Brauer graph, where $m(v_L^{(i)}) = m(v_L^{(j)}) = 2$ and $m(v) = 1$ for the other vertices v. Let B be the Brauer graph algebra associated with the new Brauer graph G' and $\overline{B}$ the corresponding string algebra defined in (2.1). Note that the quiver of $\overline{B}$ is also Q. By the construction of the Brauer graph G' and Theorem 2.9, we have that B and $\overline{B}$ are 1-domestic.

Let $E_4 = \{i_k | \mathrm{grd}(v_L^{(i)}) \ge \mathrm{grd}(v'_k), 1 < \mathrm{val}(v'_k), 2 \le k \le t\}$ and $E_5 = \{j_k | \mathrm{grd}(v_L^{(j)}) \ge \mathrm{grd}(w'_k), 1 < \mathrm{val}(w'_k), 2 \le k \le t_1\}$ (which are also defined in the original Brauer graph G), where the edge i_k (resp. j_k) is incident to $v_L^{(i)}$ (resp. $v_L^{(j)}$) and v'_k (resp. w'_k) in G for any $2 \le k \le t$ (resp. $2 \le k \le t_1$). We can get the following algebra isomorphism from their constructions

$$\overline{\mathrm{gr}(A)}/(\sum_{l\in E_3} kr_l) \cong \overline{B}/(\bigoplus_{k\in E_1\setminus E_4} \mathrm{rad}^{t+1}(P_k) \oplus \bigoplus_{k\in E_4} \mathrm{rad}^{t}(P_k) \oplus \bigoplus_{k\in E_2\setminus E_5} \mathrm{rad}^{t_1+1}(P_k) \oplus \bigoplus_{k\in E_5} \mathrm{rad}^{t_1}(P_k)),$$

where P_k is the projective cover of the simple $\overline{B}$-module S_k corresponding to the edge k in G'.

Since $\overline{B}$ is 1-domestic and $\overline{\mathrm{gr}(A)}/(\sum_{l\in E_3} kr_l)$ is of infinite representation type, we have that $\overline{\mathrm{gr}(A)}/(\sum_{l\in E_3} kr_l)$ is 1-domestic and therefore $\mathrm{gr}(A)$ is 1-domestic. Hence, the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is 1.

If $m_0 = 2$, $\kappa_0 = 1$ and $\kappa_1 = 0$, then there is only one unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G such that v_0 is in $G_{i,S}$. Similarly as above, let $i_1 < i_2 < \cdots < i_t < i_1$ be the cyclic ordering at v_L, where $i_1 = i$ and $t = \mathrm{grd}(v_L) = \mathrm{val}(v_L)$. Let $E_1 = \{i_1, i_2, \dots, i_t\}$, where the edge i_k is incident to v_L and v'_k for any $2 \le k \le t$. We denote by E_2 the set of all unbalanced edges different from i in G, and by r_l the element in $\mathbb{P}$ corresponding to an unbalanced edge l in E_2.

Let $G' = (V(G), E(G), m)$ be a Brauer graph, where $m(v_L) = m(v_0) = 2$ and $m(v) = 1$ for the other vertices v. Let B be the Brauer graph algebra associated with the new Brauer graph G' and $\overline{B}$ the corresponding string algebra defined in (2.1). Note that B and $\overline{B}$ are 1-domestic.

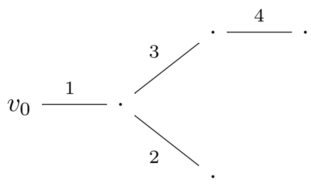
Let $E_3 = \{i_k | \mathrm{grd}(v_L) \ge \mathrm{grd}(v'_k), 1 < \mathrm{val}(v'_k), 2 \le k \le t\}$ (which is also defined in the original Brauer graph G), where the edge i_k is incident to v_L and v'_k in G for any $2 \le k \le t$. We can get the following algebra isomorphism

$$\overline{\mathrm{gr}(A)}/(\sum_{l\in E_2} kr_l) \cong \overline{B}/(\bigoplus_{k\in E_1\setminus E_3} \mathrm{rad}^{t+1}(P_k) \oplus \bigoplus_{k\in E_3} \mathrm{rad}^{t}(P_k)),$$

where P_k is the projective cover of the simple $\overline{B}$-module S_k corresponding to the edge k in G'. We also have that $\mathrm{gr}(A)$ is 1-domestic and the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is 1. $\square$

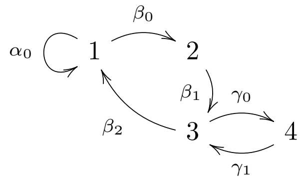
We give an example to illustrate the above result.

Example 4.6. Let G be the following Brauer tree with $m_0 = 2$.



Note that $\kappa_0 = 1$ and $\kappa_1 = 0$.

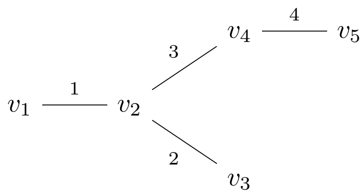
Let $A = kQ/I$ be the Brauer tree algebra associated with G and $\mathrm{gr}(A)$ the associated graded algebra of A. The quiver Q is as follows.



The regular representation of $\mathrm{gr}(A)$ is as follows.

$$\begin{array}{ccccccccccc} & 1 & & & 2 & & & 3 & & & 4 \\ 1 & & 2 & \oplus & 3 & \oplus & 1 & & 4 & \oplus & 3 \\ & & 3 & & 1 & & 2 & & & & 4 \\ & & 1 & & 2 & & 3 & & & & \end{array}$$

Note that $\mathrm{gr}(A) = \overline{\mathrm{gr}(A)}$. Moreover, $b := \alpha_0^{-1}\beta_2\beta_1\beta_0$ is the unique band in $\overline{\mathrm{gr}(A)}$ (after rotation or taking inverse). We have that G' is the following Brauer graph



where $m(v_1) = 2$, $m(v_2) = 2$ and $m(v_3) = m(v_4) = m(v_5) = 1$. The regular representation of the corresponding Brauer graph algebra B is as follows.

$$\begin{array}{ccccccccccc} & 1 & & & 2 & & & 3 & & & 4 \\ 1 & & 2 & & 3 & & 1 & & & & \\ & & 3 & & 1 & & 2 & & 4 & & \\ & & 1 & \oplus & 2 & \oplus & 3 & & & \oplus & 3 \\ & & 2 & & 3 & & 1 & & & & \\ & & 3 & & 1 & & 2 & & & & \\ & 1 & & & 2 & & & 3 & & & 4 \end{array}$$

Note that $\overline{B} = B/(\mathrm{soc}(P_1) \oplus \mathrm{soc}(P_3))$ and $\overline{\mathrm{gr}(A)}/kr_3 \cong \overline{B}/(\mathrm{rad}^4(P_1) \oplus \mathrm{rad}^4(P_2) \oplus \mathrm{rad}^3(P_3))$, where P_i is the projective cover of the simple $\overline{B}$-module S_i corresponding to the edge i in G'. Since B is 1-domestic, $\overline{B}$ is 1-domestic, and $\overline{\mathrm{gr}(A)}/kr_3$ is of infinite representation type, we have that $\overline{\mathrm{gr}(A)}/kr_3$ is 1-domestic and therefore $\mathrm{gr}(A)$ is 1-domestic.

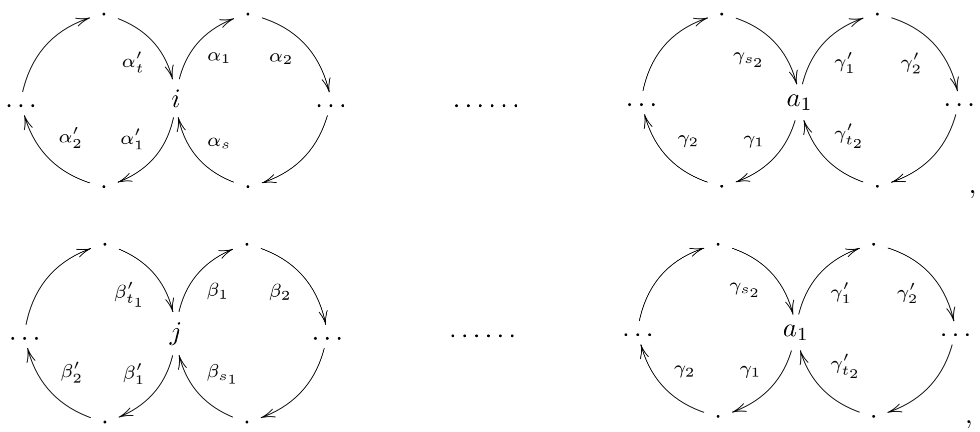
Lemma 4.7. *Let* $\overline{\mathrm{gr}(A)} = kQ/I_2$ *be defined in* (2.2). *If one of the following is satisfied:*

(1) $m_0 = 2$ *and* $\kappa_0 \geq 2$,
(2) $m_0 = 2$ *and* $\kappa_1 \neq 0$,

then the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ *is infinite and* $\mathrm{gr}(A)$ *is not of polynomial growth.*

Proof. *Case 1:* $m_0 = 2$ *and* $\kappa_0 \geq 2$. In this case there are two unbalanced edges $v_S^{(i)} \overset{i}{\text{---}} v_L^{(i)}$ and $v_S^{(j)} \overset{j}{\text{---}} v_L^{(j)}$ in G such that the exceptional vertex v_0 is a vertex in $G_{i,S}$ and it is also a vertex in $G_{j,S}$. There is a walk $[v_1, a_1, v_2, a_2, v_3, \ldots, v_{k-1}, a_{k-1}, v_k]$ (resp. $[v'_1, a'_1, v'_2, a'_2, v'_3, \ldots, v'_{k'-1}, a'_{k'-1}, v'_{k'}]$) from v_0 to $v_L^{(i)}$ (resp. $v_L^{(j)}$), where $v_1 = v_0$, $v_k = v_L^{(i)}$, $a_{k-1} = i$ (resp. $v'_1 = v_0$, $v'_{k'} = v_L^{(j)}$, $a'_{k'-1} = j$) and a_l (resp. a'_l) is an edge incident to the vertices v_l (resp. v'_l) and v_{l+1} (resp. v'_{l+1}) for each $1 \leq l \leq k-1$ (resp. $1 \leq l \leq k'-1$). We consider two subcases (a) and (b).

(a) If $a_1 = a'_1$, then Q contains the following two subquivers



where $s = \mathrm{val}(v_S^{(i)})$, $t = \mathrm{val}(v_L^{(i)})$, $s_1 = \mathrm{val}(v_S^{(j)})$, $t_1 = \mathrm{val}(v_L^{(j)})$, $s_2 = \mathrm{val}(v_2)$, $t_2 = \mathrm{val}(v_0)$, $\alpha'_t \ldots \alpha'_1$, $\beta'_{t_1} \ldots \beta'_1$ and $\gamma'_{t_2} \ldots \gamma'_1$ are not in I_2.

We will show that the two walks mentioned at the beginning of Case 1 correspond to two different bands b_1 and b_2 in $\overline{\mathrm{gr}(A)}$ that satisfy the condition of Lemma 3.13.

The band b_1 is defined as follows. There is a simple string $c_{k_1} \ldots c_2 c_1$ satisfying $c_1 = \gamma_{s_2}^{-1}$ and $t(c_{k_1}) = i$. There are two situations for c_{k_1}.

(1) If c_{k_1} is an inverse arrow (in other words, $c_{k_1} = \alpha_1^{-1}$), then $\alpha'_t \ldots \alpha'_2 \alpha'_1 c_{k_1} \ldots c_2 c_1 \gamma'_{t_2} \cdots \gamma'_1$ is also a string. There exists a simple string $c'_{k_2} \ldots c'_2 c'_1$ satisfying $c'_1 = \alpha_s^{-1}$ and $t(c'_{k_2}) = a_1$. Then

$$b_1 := c'_{k_2} \ldots c'_2 c'_1 \alpha'_t \ldots \alpha'_2 \alpha'_1 c_{k_1} \ldots c_2 c_1 \gamma'_{t_2} \cdots \gamma'_1$$

is a band with source a_1.

(2) If c_{k_1} is an arrow (in other words, $c_{k_1} = \alpha_s$), then $(\alpha'_1)^{-1} \ldots (\alpha'_t)^{-1} c_{k_1} \ldots c_2 c_1 \gamma'_{t_2} \ldots \gamma'_1$ is also a string. In this situation we can similarly get a band b_1 as in (1).

The band b_2 is defined as follows. There is a simple string $d_{k'_1} \ldots d_2 d_1$ satisfying $d_1 = \gamma_{s_2}^{-1}$ and $t(d_{k'_1}) = j$. Similarly, we have two situations for $d_{k'_1}$. Then $b_2 := d'_{k'_2} \ldots d'_2 d_1 \beta'_{t_1} \ldots \beta'_1 d_{k'_1} \ldots d_2 d_1 \gamma'_{t_2} \cdots \gamma'_1$ (or $b_2 := d'_{k'_2} \ldots d'_2 d'_1 (\beta'_1)^{-1} \ldots (\beta'_{t_1})^{-1} d_{k'_1} \ldots d_2 d_1 \gamma'_{t_2} \cdots \gamma'_1$) is a band with source a_1.

(b) If $a_1 \neq a'_1$, then $d_G(v_S^{(j)}, v_S^{(i)}) + 1 = d_G(v_L^{(j)}, v_S^{(i)})$ and j is in $G_{i,S}$. Therefore $\{i, j\}$ is an unbalanced edge pair. We have that Q contains the following subquivers

,

,

where $s = \mathrm{val}(v_S^{(i)})$, $t = \mathrm{val}(v_L^{(i)})$, $s_1 = \mathrm{val}(v_S^{(j)})$, $t_1 = \mathrm{val}(v_L^{(j)})$, $t_2 = \mathrm{val}(v_0)$, $s_2 = \mathrm{val}(v_2)$, $\alpha'_t \dots \alpha'_1$, $\beta'_{t_1} \dots \beta'_1$ and $\gamma'_{t_2} \dots \gamma'_1$ are not in I_2.

We will also show that in this case there are two different bands b_1 and b_2 in $\overline{\mathrm{gr}(A)}$ that satisfy the condition of Lemma 3.13.

The band b_1 is defined as follows. There is a simple string $c_{k_1} \dots c_2 c_1$ satisfying $c_1 = \alpha_s^{-1}$ and $t(c_{k_1}) = j$. There are two situations for c_{k_1}.

(1) If c_{k_1} is an inverse arrow (in other words, $c_{k_1} = \beta_1^{-1}$), then $\beta'_{t_1} \dots \beta'_1 c_{k_1} \dots c_2 c_1 \alpha'_t \cdots \alpha'_1$ is also a string. There exists a simple string $c'_{k_2} \dots c'_2 c'_1$ satisfying $c'_1 = \beta_{s_1}^{-1}$ and $t(c'_{k_2}) = i$. Then

$$b_1 := c'_{k_2} \dots c'_2 c'_1 \beta'_{t_1} \dots \beta'_1 c_{k_1} \dots c_2 c_1 \alpha'_t \cdots \alpha'_1$$

is a band with source i.

(2) If c_{k_1} is an arrow (in other words, $c_{k_1} = \beta_{s_1}$), then $(\beta'_1)^{-1} \dots (\beta'_{t_1})^{-1} c_{k_1} \dots c_2 c_1 \alpha'_t \cdots \alpha'_1$ is also a string. In this situation we can similarly get a band b_1 as in (1).

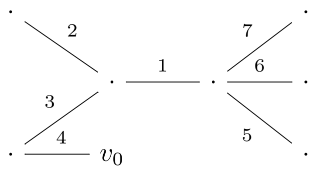
The band b_2 is defined as follows. There is a simple string $d_{k'_1} \dots d_2 d_1$ satisfying $d_1 = \alpha_s^{-1}$ and $t(d_{k'_1}) = a_1$. Similarly, we have two situations for $d_{k'_1}$. Then $b_2 := d'_{k'_2} \dots d'_2 d'_1 \gamma'_{t_2} \dots \gamma'_2 \gamma'_1 d_{k'_1} \dots d_2 d_1 \alpha'_t \cdots \alpha'_1$ (or $b_2 := d'_{k'_2} \dots d'_2 d'_1 (\gamma'_1)^{-1} \dots (\gamma'_{t_2})^{-1} d_{k'_1} \dots d_2 d_1 \alpha'_t \cdots \alpha'_1$) is a band with source i.

In either case of (a) and (b), we have that two distinct bands b_1 and b_2 in $\overline{\mathrm{gr}(A)}$ and b_1 and b_2 satisfy the condition of Lemma 3.13 by construction. Therefore, the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is infinite and $\mathrm{gr}(A)$ is not of polynomial growth.

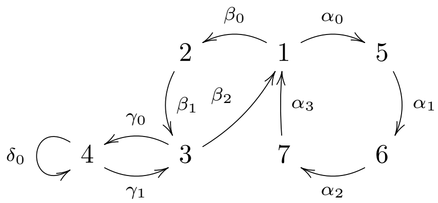
Case 2. If $m_0 = 2$ and $\kappa_1 \neq 0$, by Lemma 3.16, then there are two unbalanced edges $v_S^{(i)} \overset{i}{\text{——}} v_L^{(i)}$ and $v_S^{(j)} \overset{j}{\text{——}} v_L^{(j)}$ in G such that v_0 is in $G_{i,S}$ and $\{i, j\}$ is an unbalanced edge pair. It is similar to the above case (b). We still get our desired result. $\square$

We give an example to illustrate the above result.

Example 4.8. Let G be the following Brauer tree with $m_0 = 2$.



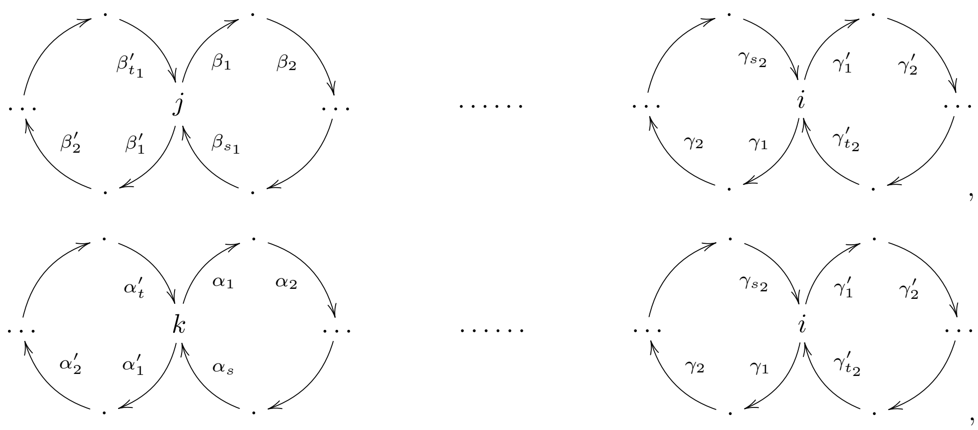
Let $A = kQ/I$ be the Brauer graph algebra associated with G and $\mathrm{gr}(A)$ the associated graded algebra of A. The quiver Q is as follows.



We have that $b_1 = \gamma_1^{-1}\beta_1\beta_0\beta_2\gamma_0^{-1}\delta_0$ and $b_2 = \gamma_1^{-1}\beta_1\beta_0\alpha_0^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1}\beta_2\gamma_0^{-1}\delta_0$ are bands in $\overline{\mathrm{gr}(A)}$.

Lemma 4.9. *Let* $\overline{\mathrm{gr}(A)} = kQ/I_2$ *be defined in* (2.2). *If* $\kappa_1 \geq 2$, *then the cardinality of* $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ *is infinite and* $\mathrm{gr}(A)$ *is not of polynomial growth.*

Proof. Since $\kappa_1 \geq 2$, by Lemma 3.17, there are three unbalanced edges $v_S^{(i)} \overset{i}{\text{——}} v_L^{(i)}$, $v_S^{(j)} \overset{j}{\text{——}} v_L^{(j)}$ and $v_S^{(k)} \overset{k}{\text{——}} v_L^{(k)}$ in G such that $\{i, j\}$ and $\{i, k\}$ are unbalanced edge pairs. Then Q contains the following two subquivers



where $s = \mathrm{val}(v_S^{(k)})$, $t = \mathrm{val}(v_L^{(k)})$, $s_1 = \mathrm{val}(v_S^{(j)})$, $t_1 = \mathrm{val}(v_L^{(j)})$, $s_2 = \mathrm{val}(v_S^{(i)})$, $t_2 = \mathrm{val}(v_L^{(i)})$, $\alpha'_t \ldots \alpha'_1$, $\beta'_{t_1} \ldots \beta'_1$ and $\gamma'_{t_2} \ldots \gamma'_1$ are not in I_2.

We will show that there are two different bands b_1 and b_2 in $\overline{\mathrm{gr}(A)}$ that satisfy the condition of Lemma 3.13.

The band b_1 is defined as follows. There is a simple string $c_{k_1} \ldots c_2 c_1$ satisfying $c_1 = \gamma_{s_2}^{-1}$ and $t(c_{k_1}) = j$. There are two situations for c_{k_1}.

(1) If c_{k_1} is an inverse arrow (in other words, $c_{k_1} = \beta_1^{-1}$), then $\beta'_{t_1} \ldots \beta'_1 c_{k_1} \ldots c_2 c_1 \gamma'_{t_2} \cdots \gamma'_1$ is also a string. There exists a simple string $c'_{k_2} \ldots c'_2 c'_1$ satisfying $c'_1 = \beta_{s_1}^{-1}$ and $t(c'_{k_2}) = i$. Then

$$b_1 := c'_{k_2} \dots c'_2 c'_1 \beta'_{t_1} \dots \beta'_1 c_{k_1} \dots c_2 c_1 \gamma'_{t_2} \cdots \gamma'_1$$

is a band with source i.

(2) If c_{k_1} is an arrow (in other words, $c_{k_1} = \beta_{s_1}$), then $(\beta'_1)^{-1} \dots (\beta'_{t_1})^{-1} c_{k_1} \dots c_2 c_1 \gamma'_{t_2} \dots \gamma'_1$ is also a string. In this situation we can similarly get a band b_1 as in (1).

The band b_2 is defined as follows. There is a simple string $d_{k'_1} \dots d_2 d_1$ satisfying $d_1 = \gamma_{s_2}^{-1}$ and $t(d_{k'_1}) = k$. Similarly, we have two situations for $d_{k'_1}$. Then $b_2 := d'_{k'_2} \dots d'_2 d'_1 \alpha'_t \dots \alpha'_2 \alpha'_1 d_{k'_1} \dots d_2 d_1 \gamma'_{t_2} \cdots \gamma'_1$ (or $b_2 := d'_{k'_2} \dots d'_2 d'_1 (\alpha'_1)^{-1} \dots (\alpha'_t)^{-1} d_{k'_1} \dots d_2 d_1 \gamma'_{t_2} \cdots \gamma'_1$) is a band with source i.

Moreover, b_1 and b_2 satisfy the condition of Lemma 3.13 by construction. Therefore, the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is infinite and $\mathrm{gr}(A)$ is not of polynomial growth. □

By the above results, we have the following characterization of domestic representation type of $\mathrm{gr}(A)$.

Proposition 4.10. *Let A be the Brauer tree algebra associated with a Brauer tree with an exceptional vertex v_0 of multiplicity m_0 and $\mathrm{gr}(A)$ the graded algebra associated with the radical filtration of A. Then the following are equivalent.*

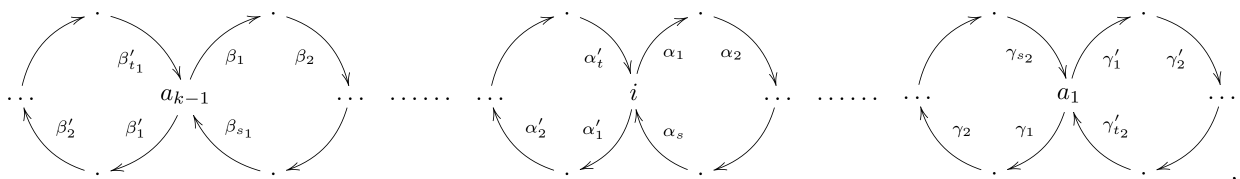
(1) $\mathrm{gr}(A)$ *is of polynomial growth.*
(2) $\mathrm{gr}(A)$ *is domestic.*
(3) $\mathrm{gr}(A)$ *is 1-domestic.*
(4) $\kappa_0(m_0 - 1) + \kappa_1 = 1$.
(5) *The cardinality of* $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ *is finite, where* $\overline{\mathrm{gr}(A)}$ *is defined in* (2.2).

5. The case that G is a tree with $m(v) = 2$ for exactly two vertices and $m(v) = 1$ for other vertices

In this section, we describe when $\mathrm{gr}(A) = kQ/I'$ is domestic under the assumption that $G = (V(G), E(G), m)$ is a tree with $m(v) = 2$ for exactly two vertices $v = w_0, w_1 \in V(G)$ and $m(v) = 1$ for all $v \in V(G)$, $v \neq w_0, w_1$, where $V(G)$ is the vertex set, $E(G)$ is the edge set and m is the multiplicity function of G.

Lemma 5.1. *Let $\overline{\mathrm{gr}(A)} = kQ/I_2$ be defined in* (2.2). *Suppose that there is an unbalanced edge $v_S \xrightarrow{\ i\ } v_L$ in G such that w_0 and w_1 are in different connected branch of $G \setminus i$. Then the cardinality of* $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ *is infinite and* $\mathrm{gr}(A)$ *is not of polynomial growth.*

Proof. Without loss of generality, we assume that w_0 is in $G_{i,S}$ and w_1 is in $G_{i,L}$. We consider the walk $[v_1, a_1, v_2, \dots, v_{k-1}, a_{k-1}, v_k]$ from w_0 to w_1, where $v_1 = w_0$, $v_k = w_1$. There is an edge $v_j \xrightarrow{a_j} v_{j+1}$ in the walk such that $a_j = i$, $v_j = v_S$ and $v_{j+1} = v_L$. Then Q contains the following subquiver



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where $s = \mathrm{val}(v_S)$, $t = \mathrm{val}(v_L)$, $s_1 = \mathrm{val}(v_{k-1})$, $t_1 = \mathrm{val}(w_1)$, $s_2 = \mathrm{val}(v_2)$, $t_2 = \mathrm{val}(w_0)$, $\alpha'_t \dots \alpha'_1$, $\beta'_{t_1} \dots \beta'_1$ and $\gamma'_{t_2} \dots \gamma'_1$ are not in I_2.

We will show that there are two different bands b_1 and b_2 in $\overline{\mathrm{gr}(A)}$ that satisfy the condition of Lemma 3.13.

The band b_1 is defined as follows. There is a simple string $c_{k_1} \dots c_2 c_1$ satisfying $c_1 = \gamma_{s_2}^{-1}$ and $t(c_{k_1}) = a_{k-1}$. There are two situations for c_{k_1}.

(1) If c_{k_1} is an inverse arrow (in other words, $c_{k_1} = \beta_1^{-1}$), then $\beta'_{t_1} \dots \beta'_1 c_{k_1} \dots c_2 c_1 \gamma'_{t_2} \cdots \gamma'_1$ is also a string. There exists a simple string $c'_{k_2} \dots c'_2 c'_1$ satisfying $c'_1 = \beta_{s_1}^{-1}$ and $t(c'_{k_2}) = a_1$. Then

$$b_1 := c'_{k_2} \dots c'_2 c'_1 \beta'_{t_1} \dots \beta'_1 c_{k_1} \dots c_2 c_1 \gamma'_{t_2} \cdots \gamma'_1$$

is a band with source a_1.

(2) If c_{k_1} is an arrow (in other words, $c_{k_1} = \beta_{s_1}$), then $(\beta'_1)^{-1} \dots (\beta'_{t_1})^{-1} c_{k_1} \dots c_2 c_1 \gamma'_{t_2} \dots \gamma'_1$ is also a string. In this situation we can similarly get a band b_1 as in (1).

The band b_2 is defined as follows. There is a simple string $d_{k'_1} \dots d_2 d_1$ satisfying $d_1 = \gamma_{s_2}^{-1}$ and $t(d_{k'_1}) = i$. Similarly, we have two situations for $d_{k'_1}$. Then $b_2 := d'_{k'_2} \dots d'_2 d'_1 \alpha'_t \dots \alpha'_2 \alpha'_1 d_{k'_1} \dots d_2 d_1 \gamma'_{t_2} \cdots \gamma'_1$ (or $b_2 := d'_{k'_2} \dots d'_2 d'_1 (\alpha'_1)^{-1} \dots (\alpha'_t)^{-1} d_{k'_1} \dots d_2 d_1 \gamma'_{t_2} \cdots \gamma'_1$) is a band with source a_1.

Moreover, b_1 and b_2 satisfy the condition of Lemma 3.13 by construction. Therefore, the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is infinite and $\mathrm{gr}(A)$ is not of polynomial growth. □

Proposition 5.2. *Let $\overline{\mathrm{gr}(A)} = kQ/I_2$ be defined in (2.2). If there is an unbalanced edge $v_S \overset{i}{\text{---}} v_L$ in G such that w_0 and w_1 are in $G_{i,S}$, then $\overline{\mathrm{gr}(A)}$ and $\mathrm{gr}(A)$ are not of polynomial growth.*

Proof. This can be proved by constructing infinitely many bands in $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ using the similar method in Lemma 5.1. Alternatively, we can use an approach similar to the proof of Proposition 4.3 to get a quotient algebra C of $\overline{\mathrm{gr}(A)}$ which is a representation-infinite string algebra and is not of polynomial growth. Therefore we have that $\overline{\mathrm{gr}(A)}$ and $\mathrm{gr}(A)$ are not of polynomial growth. □

By Proposition 3.11, Lemma 5.1 and Proposition 5.2, we have the following characterization of domestic representation type of $\mathrm{gr}(A)$.

Proposition 5.3. *Let A be the Brauer graph algebra associated with a Brauer graph G which is a tree with $m(v) = 2$ for exactly two vertices $v = w_0, w_1$ and $m(v) = 1$ for all $v \neq w_0, w_1$, and $\mathrm{gr}(A)$ the graded algebra associated with the radical filtration of A. Then the following are equivalent.*

(1) $\mathrm{gr}(A)$ *is of polynomial growth.*
(2) $\mathrm{gr}(A)$ *is domestic.*
(3) $\mathrm{gr}(A)$ *is 1-domestic.*
(4) *There is no unbalanced edge in G or w_0 and w_1 are in $G_{i,L}$ for any unbalanced edge $v_S \overset{i}{\text{---}} v_L$ in G (In other words, G satisfies $\star$-condition with respect to any unbalanced edge in G).*
(5) *The cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is finite, where $\overline{\mathrm{gr}(A)}$ is defined in (2.2).*

We can describe when $\mathrm{gr}(A)$ is domestic from the graded degrees of vertices in G point of view in the following

Proposition 5.4. *Let A be the Brauer graph algebra associated with a Brauer graph G which is a tree with $m(v) = 2$ for exactly two vertices $v = w_0, w_1$ and $m(v) = 1$ for all $v \neq w_0, w_1$, and $\mathrm{gr}(A)$ the associated graded algebra of A. Then $\mathrm{gr}(A)$ is domestic if and only if it satisfies the following conditions.*

(1) $\mathrm{grd}(w_0) = \mathrm{grd}(w_1)$.
(2) *Any walk from w_0 (or from w_1) is degree decreasing.*

Proof. "$\Longrightarrow$" Suppose on the contrary that $\mathrm{grd}(w_0) \neq \mathrm{grd}(w_1)$. Consider the walk $[v_1, a_1, v_2, \dots, v_{k-1}, a_{k-1}, v_k]$ from w_0 to w_1, where $v_1 = w_0$, $v_k = w_1$, we have that there is an unbalanced edge $v_i \xrightarrow{a_i} v_{i+1}$ with $\mathrm{grd}(v_i) \neq \mathrm{grd}(v_{i+1})$ for some $1 \le i \le k-1$ in the walk. Without loss of generality, we assume that $\mathrm{grd}(v_i) < \mathrm{grd}(v_{i+1})$. Then w_0 is in $G_{a_i,S}$ and w_1 is in $G_{a_i,L}$, by Proposition 5.3, $\mathrm{gr}(A)$ is nondomestic which is a contradiction. Therefore $\mathrm{grd}(w_0) = \mathrm{grd}(w_1)$ and the condition (1) holds.

In order to verify the condition (2). Suppose on the contrary that there is a vertex w' in G such that the walk $[v_1, a_1, v_2, a_2, v_3, \dots, v_{k-1}, a_{k-1}, v_k]$ from w_0 to w' is not degree decreasing, where $v_1 = w_0$ and $v_k = w'$. In other words, there exists an unbalanced edge $v_i \xrightarrow{a_i} v_{i+1}$ with $\mathrm{grd}(v_i) < \mathrm{grd}(v_{i+1})$ for some $1 \le i \le k-1$ in the walk. We have that w_0 is in $G_{a_i,S}$. Moreover, since $\mathrm{gr}(A)$ is domestic, by Proposition 5.3, w_0 is in $G_{a_i,L}$. A contradiction.

"$\Longleftarrow$" We suppose on the contrary that $\mathrm{gr}(A)$ is nondomestic. By Proposition 2.11 and Proposition 5.3, we have that there is some unbalanced edge $v_S \xrightarrow{i} v_L$ such that w_0, w_1 are in $G_{i,S}$ or w_0, w_1 are in different connected branch of $G \setminus i$.

Case 1. If w_0, w_1 are in $G_{i,S}$. Consider the walk $[v_1, a_1, v_2, \dots, v_{k-1}, a_{k-1}, v_k]$ from w_0 to v_L, where $v_1 = w_0$ and $v_k = v_L$. Then $i = a_{k-1}$. Since the above walk is degree decreasing, we have $\mathrm{grd}(v_S) \ge \mathrm{grd}(v_L)$, which is clearly a contradiction.

Case 2. If w_0, w_1 are in different connected branch of $G \setminus i$. Consider the walk $[v_1, a_1, v_2, \dots, v_{k-1}, a_{k-1}, v_k]$ from w_0 to w_1, where $v_1 = w_0$ and $v_k = w_1$. Then i is an edge in the walk. Since the above walk is degree decreasing, we have $\mathrm{grd}(w_0) \neq \mathrm{grd}(w_1)$. It contradicts the condition (1). □

6. The case that G is a graph with a unique cycle and $m \equiv 1$

In this section, we describe when $\mathrm{gr}(A) = kQ/I'$ is domestic in the case that G is a graph with a unique cycle and $m(v) = 1$ for any vertex v in G.

Lemma 6.1. *The number of unbalanced edges in the unique cycle is always strictly greater than* 1 *if it is non-zero. Precisely, if there is an unbalanced edge $v_S^{(i)} \xrightarrow{i} v_L^{(i)}$ in the unique cycle, then there is another unbalanced edge $v_S^{(j)} \xrightarrow{j} v_L^{(j)}$ with $v_S^{(i)} \neq v_L^{(j)}$ in the unique cycle such that there is a walk $[v_1, a_1, v_2, \dots, v_{k-1}, a_{k-1}, v_k]$ from $v_S^{(i)}$ to $v_L^{(j)}$ satisfying $i \neq a_1$ and $a_{k-1} = j$, where $v_1 = v_S^{(i)}$, $v_k = v_L^{(j)}$ and a_l is an edge in the unique cycle incident to the vertices v_l and v_{l+1} for each $1 \le l \le k-1$.*

Proof. Note that the unique cycle is connected, if there exists an unbalanced edge in the unique cycle, then the number of unbalanced edges in the unique cycle is strictly greater than 1.

For any unbalanced edge $v_S^{(i)} \xrightarrow{i} v_L^{(i)}$ in the unique cycle, there is also an unbalanced edge $v_S^{(j)} \xrightarrow{j} v_L^{(j)}$ different from i in the unique cycle. We can assume that $v_S^{(i)} \neq v_L^{(j)}$; indeed, if $v_S^{(i)} = v_L^{(j)}$, then $\mathrm{grd}(v_S^{(j)}) < \mathrm{grd}(v_L^{(j)}) = \mathrm{grd}(v_S^{(i)}) < \mathrm{grd}(v_L^{(i)})$, and by the connectivity of cycle, there is another unbalanced edge $v_S^{(j')} \xrightarrow{j'} v_L^{(j')}$ (which is different from i and j) in the unique cycle with $v_S^{(i)} \neq v_L^{(j')}$.

The above shows that for any unbalanced edge $v_S^{(i)} \xrightarrow{i} v_L^{(i)}$ in the unique cycle, we have another unbalanced edge $v_S^{(j)} \xrightarrow{j} v_L^{(j)}$ satisfying $v_S^{(i)} \neq v_L^{(j)}$ in the unique cycle. There is a walk $[v_1, a_1, v_2, \dots, v_{k-1}, a_{k-1}, v_k]$ from $v_S^{(i)}$ to $v_L^{(j)}$ satisfying $i \neq a_1$, where $v_1 = v_S^{(i)}$, $v_k = v_L^{(j)}$ and a_l is an edge in the unique cycle incident to the vertices v_l and v_{l+1} for each $1 \le l \le k-1$. Moreover, there is also a walk $[v'_1, a'_1, v'_2, \dots, v'_{k'-1}, a'_{k'-1}, v'_{k'}]$ from $v_S^{(i)}$ to $v_L^{(j)}$ different from the above walk, where $v'_1 = v_S^{(i)}$, $v'_{k'} = v_L^{(j)}$, $a'_1 = i$ and a'_l is an edge in the unique cycle incident to the vertices v'_l and v'_{l+1} for each $1 \le l \le k'-1$. We have the following two cases for a_{k-1}.

(*a*) If $a_{k-1} = j$, then j and the walk $[v_1, a_1, v_2, \dots, v_{k-1}, a_{k-1}, v_k]$ give our desired result.
(*b*) If $a_{k-1} \neq j$, then $a'_{k'-1} = j$ and there are two cases to be considered.

(1) If $\mathrm{grd}(v_S^{(j)}) < \mathrm{grd}(v_L^{(i)})$, by the connectivity of cycle, then there is an unbalanced edge a'_t in the walk $[a'_1, a'_2, \ldots, a'_{k'-1}]$ satisfying $\mathrm{grd}(v'_t) > \mathrm{grd}(v'_{t+1})$ and $v'_t \neq v_S^{(i)}$. The unbalanced edge a'_t and the walk $[v_1, a_1, v_2, \ldots, v_k, j, v_S^{(j)}, \ldots, v'_{t+1}, a'_t, v'_t]$ give our desired result.

(2) If $\mathrm{grd}(v_S^{(j)}) \geq \mathrm{grd}(v_L^{(i)})$, then $\mathrm{grd}(v_S^{(i)}) < \mathrm{grd}(v_L^{(j)})$ and there is an unbalanced edge a_t in the walk $[a_1, a_2, \ldots, a_{k-1}]$ satisfying $\mathrm{grd}(v_t) < \mathrm{grd}(v_{t+1})$. Therefore the unbalanced edge a_t and the walk $[v_1, a_1, v_2, \ldots, v_t, a_t, v_{t+1}]$ give our desired result. □

Lemma 6.2. *Let* $\overline{\mathrm{gr}(A)} = kQ/I_2$ *be defined in* (2.2). *If some edges in the unique cycle are unbalanced edges, then the cardinality of* $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ *is infinite and* $\mathrm{gr}(A)$ *is not of polynomial growth.*

Proof. If some edges in the unique cycle are unbalanced edges, by Lemma 6.1, then there are at least two unbalanced edges $v_S^{(i)} \overset{i}{\text{---}} v_L^{(i)}$ and $v_S^{(j)} \overset{j}{\text{---}} v_L^{(j)}$ with $v_S^{(i)} \neq v_L^{(j)}$ in the unique cycle such that there is a walk $[v_1, a_1, v_2, a_2, v_3, \ldots, v_{k-1}, a_{k-1}, v_k]$ from $v_S^{(i)}$ to $v_L^{(j)}$ satisfying $i \neq a_1$ and $a_{k-1} = j$, where $v_1 = v_S^{(i)}$, $v_k = v_L^{(j)}$ and a_l is an edge in the unique cycle incident to the vertices v_l and v_{l+1} for each $1 \leq l \leq k-1$. Then Q contains the following subquiver

where $s = \mathrm{val}(v_S^{(i)})$, $t = \mathrm{val}(v_L^{(i)})$, $s_1 = \mathrm{val}(v_S^{(j)})$, $t_1 = \mathrm{val}(v_L^{(j)})$, $\alpha'_t \ldots \alpha'_1$ and $\beta'_{t_1} \ldots \beta'_1$ are not in I_2.

Since there is a unique cycle in G, there is a band $b_1 = \alpha'_t \ldots \alpha_s^{-1}$ in $\overline{A}$. Therefore b_1 is also a band in $\overline{\mathrm{gr}(A)}$.

Next we construct another band b_2 in $\overline{\mathrm{gr}(A)}$. There is a simple string $c_{k_1} \ldots c_2 c_1$ satisfying $c_1 = \alpha_s^{-1}$ and $t(c_{k_1}) = j$ and it is constructed from the walk $[a_1, a_2, \ldots, a_{k-1}]$. There are two situations.

(1) If c_{k_1} is an inverse arrow (in other words, $c_{k_1} = \beta_1^{-1}$), then $\beta'_{t_1} \ldots \beta'_1 c_{k_1} \ldots c_2 c_1 \alpha'_t \cdots \alpha'_1$ is also a string. There exists a simple string $c'_{k_2} \ldots c'_2 c'_1$ satisfying $c'_1 = \beta_{s_1}^{-1}$ and $t(c'_{k_2}) = i$. Then

$$b_2 := \alpha'_t \ldots \alpha'_1 c'_{k_2} \ldots c'_2 c'_1 \beta'_{t_1} \ldots \beta'_1 c_{k_1} \ldots c_2 c_1$$

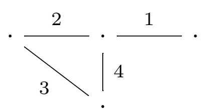
is a band with source i.

(2) If c_{k_1} is an arrow (in other words, $c_{k_1} = \beta_{s_1}$), then $(\beta'_1)^{-1} \ldots (\beta'_{t_1})^{-1} c_{k_1} \ldots c_2 c_1\ \alpha'_t \ldots \alpha'_1$ is also a string. In this situation we can similarly get a band b_2 as in (1).

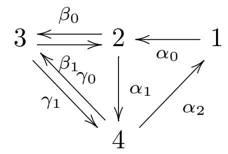
Using an approach similar to the proof of Lemma 3.13, we have that the cardinality of $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ is infinite and $\mathrm{gr}(A)$ is not of polynomial growth. □

We give an example to illustrate the above result.

Example 6.3. Let G be the following Brauer graph with $m \equiv 1$.



Let $A = kQ/I$ be the Brauer graph algebra associated with G and $\mathrm{gr}(A)$ the associated graded algebra of A. The quiver Q is as follows.



Note that $\overline{\mathrm{gr}(A)} = \mathrm{gr}(A)/\operatorname{soc}(P_3)$, where P_3 is the projective cover of simple $\mathrm{gr}(A)$-module S_3 corresponding to the vertex 3 in Q.

We have that $b_1 = \alpha_0\alpha_2\gamma_0^{-1}\beta_0\alpha_1^{-1}\gamma_1\beta_1^{-1}$ and $b_2 = \alpha_0\alpha_2\alpha_1\beta_0^{-1}\gamma_0\alpha_2^{-1}\alpha_0^{-1}\alpha_1^{-1}\gamma_1\beta_1^{-1}$ are bands in $\overline{\mathrm{gr}(A)}$.

Proposition 6.4. *Let $\overline{\mathrm{gr}(A)} = kQ/I_2$ be defined in (2.2). Suppose that there is an unbalanced edge $v_S \overset{i}{\text{---}} v_L$ which is not an edge in the unique cycle such that the unique cycle is in $G_{i,S}$. Then $\overline{\mathrm{gr}(A)}$ and $\mathrm{gr}(A)$ are not of polynomial growth.*

Proof. This can be proved by constructing infinitely many bands in $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ using the similar method in Lemma 6.2. Alternatively, we can use an approach similar to the proof of Proposition 4.3 to get a quotient algebra C of $\overline{\mathrm{gr}(A)}$ which is a representation-infinite string algebra and is not of polynomial growth. Therefore we have that $\overline{\mathrm{gr}(A)}$ and $\mathrm{gr}(A)$ are not of polynomial growth. □

By Proposition 3.12, Lemma 6.2 and Proposition 6.4, we have the following characterization of domestic representation type of $\mathrm{gr}(A)$.

Proposition 6.5. *Let A be the Brauer graph algebra associated with a Brauer graph G and $\mathrm{gr}(A)$ the graded algebra associated with the radical filtration of A, where G is a graph with a unique cycle and $m(v) = 1$ for all $v \in V(G)$. Then the following are equivalent.*

(1) $\mathrm{gr}(A)$ *is of polynomial growth.*
(2) $\mathrm{gr}(A)$ *is domestic.*
(3) $\mathrm{gr}(A)$ *is* 1*-domestic (resp.* 2*-domestic) if the unique cycle is of odd length (resp. even length).*
(4) *There is no unbalanced edges in G or all edges in the unique cycle are not unbalanced edges and the unique cycle is in $G_{i,L}$ for any unbalanced edge $v_S \overset{i}{\text{---}} v_L$ (In other words, G satisfies $\star$-condition with respect to any unbalanced edge in G).*
(5) *The cardinality of* $\mathrm{Ba}(\overline{\mathrm{gr}(A)})$ *is finite, where* $\overline{\mathrm{gr}(A)}$ *is defined in* (2.2).

We can describe when $\mathrm{gr}(A)$ is domestic from the graded degrees of vertices in G point of view in the following

Proposition 6.6. *Let A be the Brauer graph algebra associated with a Brauer graph G and $\mathrm{gr}(A)$ the associated graded algebra of A, where G is a graph with a unique cycle and $m \equiv 1$. Then $\mathrm{gr}(A)$ is domestic if and only if it satisfies the following conditions.*

(1) $\mathrm{grd}(u) = \mathrm{grd}(v)$ *for any two distinct vertices u and v in the unique cycle.*
(2) *Any walk from any vertex in the unique cycle is degree decreasing.*

Proof. "$\Longrightarrow$" Since all edges in the unique cycle are not unbalanced edges, $\mathrm{grd}(u) = \mathrm{grd}(v)$ for any two vertices u and v in the unique cycle (hence the condition (1) holds).

In order to verify the condition (2). We suppose, on the contrary that, there is a vertex w in G such that a walk $[v_1, a_1, v_2, \ldots, v_{k-1}, a_{k-1}, v_k]$ from v to w is not degree decreasing, where v is a vertex in the unique cycle and $v_k = w$. In other words, there is an unbalanced edge $v_i \xrightarrow{a_i} v_{i+1}$ with $\mathrm{grd}(v_i) < \mathrm{grd}(v_{i+1})$ for some $1 \leq i \leq k-1$ in the walk. We have v is in $G_{a_i,S}$ and the unique cycle is in $G_{a_i,S}$. Moreover, since $\mathrm{gr}(A)$ is domestic, by Proposition 6.5, the unique cycle is in $G_{a_i,L}$. A contradiction.

"$\Longleftarrow$" We suppose on the contrary that $\mathrm{gr}(A)$ is nondomestic. By Proposition 2.11 and Proposition 6.5, since it contradicts the condition (1) that some edges in the unique cycle are unbalanced edges, we have that all edges in the unique cycle are not unbalanced edges and therefore there is some unbalanced edge $v_S \xrightarrow{i} v_L$ such that the unique cycle is in $G_{i,S}$. For a vertex v in the unique cycle and any walk $[v_1, a_1, v_2, \ldots, v_{k-1}, a_{k-1}, v_k]$ from v to v_L which is degree decreasing, we have that i is an edge in the walk and therefore $\mathrm{grd}(v_S) \geq \mathrm{grd}(v_L)$, which is clearly a contradiction. Our assumption is false and therefore $\mathrm{gr}(A)$ is domestic. □

CRediT authorship contribution statement

Jing Guo: Writing – review & editing, Writing – original draft, Funding acquisition, Formal analysis, Conceptualization. **Yuming Liu:** Writing – review & editing, Writing – original draft, Funding acquisition, Formal analysis, Conceptualization. **Yu Ye:** Writing – review & editing, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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